ON THE CLASS OF COMPLETELY MONOTONIC SEQUENCES

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Abstract. In this review article, we study the recent investigations on completely monotonic sequence and the relationship among them.

1. RECENT DEVELOPMENTS ON COMPLETELY MONOTONIC AND RELATED SEQUENCES

We first introduce the notion of a completely monotonic sequence.

Definition 1 ([1]). A sequence \( \{\mu_n\}_{n=0}^{\infty} \) is called completely monotonic if

\[
(-1)^k \Delta^k \mu_n \leq 0, \quad n, k \in \mathbb{N}_0,
\]

where

\[
\Delta^0 \mu_n = \mu_n
\]

and

\[
\Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n.
\]

Here in Definition 1 and throughout the paper

\[ \mathbb{N}_0 := \{0\} \cup \mathbb{N} \]

and \( \mathbb{N} \) is the set of all positive integers.

It is clear that a completely monotonic sequence is non-negative and non-increasing.

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From Definition 1, using mathematical induction, we can prove, for a completely monotonic sequence \( \{\mu_n\}_{n=0}^\infty \), that the sequence
\[
\{(-1)^m \Delta^m \mu_n\}_{m=0}^\infty
\]
is non-negative and non-increasing for any fixed \( m \in \mathbb{N}_0 \) and that the sequence
\[
\{(-1)^m \Delta^m \mu_n\}_{m=0}^\infty
\]
is non-negative and non-increasing for any fixed \( n \in \mathbb{N}_0 \).

In [2] the authors showed that for a completely monotonic sequence \( \{\mu_n\}_{n=0}^\infty \), we always have
\[
(-1)^k \Delta^k \mu_n > 0, \quad n, k \in \mathbb{N}_0
\]
unless
\[
\mu_n = c,
\]
a non-negative constant for all \( n \in \mathbb{N} \).

The following concept of a moment sequence is closely related to the notion of a completely monotonic sequence.

**Definition 2 ([3, Chapter III]).** A sequence \( \{\mu_n\}_{n=0}^\infty \) is called a moment sequence if there exists a function \( \alpha(t) \) of bounded variation on the interval \([0, 1]\) such that
\[
\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0.
\]

Let
\[
\lambda_{k,m} := \binom{k}{m} (-1)^{k-m} \Delta^k \mu_m, \quad k, m \in \mathbb{N}_0.
\]

It was shown (see [3, Chapter III]) that

**Theorem 3.** A sequence \( \{\mu_n\}_{n=0}^\infty \) is a moment sequence if and only if there exists a constant \( L \) such that
\[
\sum_{m=0}^\infty |\lambda_{k,m}| < L, \quad k \in \mathbb{N}_0.
\]

where in (7) \( \lambda_{k,m} \) is defined by (6).

For completely monotonic sequences, the following is the well-known Hausdorff’s Theorem (see [1]).
Theorem 4. A sequence \( \{\mu_n\}_{n=0}^{\infty} \) is completely monotonic if and only if there exists a non-decreasing and bounded function \( \alpha(t) \) on \([0, 1]\) such that
\[
\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0.
\] (8)

By using Hausdorff’s Theorem 4 or difference equation (3) we can prove the following property of completely monotonic sequences, which was established in [17].

Theorem 5. Suppose that the sequence \( \{\mu_n\}_{n=0}^{\infty} \) is completely monotonic. Then, for \( m, k \in \mathbb{N}_0 \),
\[
\mu_m = (-1)^{k+1} \Delta^{k+1} \mu_m + \sum_{i=0}^{k} (-1)^i \Delta^i \mu_{m+1}.
\] (9)

From Hausdorff’s Theorem 4, Definition 2, and the relationships between the functions of bounded variation and non-decreasing, bounded functions, we know that the class of completely monotonic sequences is a subset of the class of moment sequences, and that (see [3, Chapter III])

Theorem 6. A necessary and sufficient condition that the sequence \( \{\mu_n\}_{n=0}^{\infty} \) should be a moment sequence is that it should be the difference of two completely monotonic sequences.

Widder [4] defined a sub-class of the class of completely monotonic sequences as follows:

Definition 7 ( [4]). A sequence \( \{\mu_n\}_{n=0}^{\infty} \) is called minimal completely monotonic if it is completely monotonic and if it will not be completely monotonic when \( \mu_0 \) is replaced by a number less than \( \mu_0 \).

Apparently not all completely monotonic sequences are minimal completely monotonic.

For the class of minimal completely monotonic sequences, Widder [4] proved the following result:

Theorem 8. A sequence \( \{\mu_n\}_{n=0}^{\infty} \) is minimal completely monotonic if and only if there exists a non-decreasing and bounded function \( \alpha(t) \) on
with

\[ \alpha(0) = \alpha(0+) \]

such that

\[ \mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0. \]  

(10)

We also recall the notion of a completely monotonic function, which is related to that of a completely monotonic sequence.

**Definition 9** ([5]). A function \( f \) is said to be completely monotonic on an interval \( I \) if it is continuous on \( I \), has derivatives of all orders on \( I^o \) (the interior of \( I \)) and for all \( n \in \mathbb{N}_0 \)

\[ (-1)^n f^{(n)}(x) \leq 0, \quad x \in I^o. \]

(11)

The class of all completely monotonic functions on the interval \( I \) is denoted by \( CM(I) \).

For the completely monotonic functions on the interval \([0, \infty)\), Bernstein [6] established the following result known as Bernstein’s Theorem.

**Theorem 10.** A function \( f \) on the interval \([0, \infty)\) is completely monotonic if and only if there exists a bounded and non-decreasing function \( \alpha(t) \) on \([0, \infty)\) such that

\[ f(x) = \int_0^\infty e^{-xt} d\alpha(t). \]  

(12)

There is a rich literature on completely monotonic and related functions. For more recent work, see, for example, [7-16, 18-28].

There exists a close relationship between completely monotonic functions and completely monotonic sequences. For example, Widder [4] showed the following

**Theorem 11.** Suppose that \( f \in CM [a, \infty) \), then for any \( \delta \leq 0 \), the sequence \( \{f(a + n\delta)\}_{n=0}^{\infty} \) is completely monotonic.

This result was generalized in [29] as follows:

**Theorem 12.** Suppose that \( f \in CM [a, \infty) \). If the sequence \( \{\Delta x_k\}_{k=0}^{\infty} \) is completely monotonic and \( x_0 \leq a \), then the sequence \( \{f(x_k)\}_{k=0}^{\infty} \) is also completely monotonic.
For the meaning of $\Delta x_k$, $k \in N_0$ in Theorem 12, see (2) and (3).

Suppose that $f \in CM [0, \infty)$. By Theorem 11, we know that the sequence $\{f(n)\}_{n=0}^{\infty}$ is completely monotonic.

On the other hand, for any given completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$, we may ask whether there exists an interpolating function $f \in CM [0, \infty)$ such that

$$f(n) = \mu_n, \, n \in N_0.$$ 

For this interpolation question, Widder [4] established

**Theorem 13.** There exists a function $f \in CM [0, \infty)$ such that

$$f(n) = \mu_n, \, n \in N_0$$

if and only if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is minimal completely monotonic.

From Theorem 13, we see that the condition of minimal complete monotonicity is critical for a sequence $\{\mu_n\}_{n=0}^{\infty}$ to be interpolated by a completely monotonic function on the interval $[0, \infty)$.

The result below [14] deals with a general interpolation question of completely monotonic sequences by completely monotonic functions.

**Theorem 14.** Suppose that the sequence $\{\mu_n\}_{0}^{\infty}$ is completely monotonic, then for any $\varepsilon \in (0, 1)$, there exists a continuous interpolating function $f(x)$ on the interval $[0, \infty)$ such that $f|_{[0, \varepsilon]}$ and $f|_{[\varepsilon, \infty)}$ are both completely monotonic and

$$f(n) = \mu_n, \, n \in N_0.$$ 

Based on Theorem 14 the following result [14] is obtained.

**Theorem 15.** Suppose that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic. Then there exists a completely monotonic interpolating function $g(x)$ on the interval $[1, \infty)$ such that

$$g(n) = \mu_n, \, n \in N.$$ 

It should be noted that under the condition of Theorem 15 we can not guarantee that there exists a completely monotonic interpolating function $g(x)$ on the interval $[0, \infty)$ such that

$$g(n) = \mu_n, \, n \in N_0.$$
In fact, from Theorem 13 we know that if the sequence \( \{\mu_n\}_{n=0}^{\infty} \) is not minimal completely monotonic, then there does not exist a function 
\[
f \in CM[0, \infty)
\]
such that 
\[
f(n) = \mu_n, \ n \in \mathbb{N}_0.
\]

The following result [17] provides a necessary and sufficient condition for a sequence to be completely monotonic using only the properties of its own, without using the notion of Stieltjes integrals like the well-known Hausdorff’s Theorem 4 does.

**Theorem 16.** A necessary and sufficient condition for the sequence \( \{\mu_n\}_{n=0}^{\infty} \) to be completely monotonic is that the sequence \( \{\mu_n\}_{n=1}^{\infty} \) is completely monotonic, the series
\[
\sum_{j=0}^{\infty} (-1)^j \Delta \mu_1
\]
converges and
\[
\mu_0 \leq \sum_{j=0}^{\infty} (-1)^j \Delta \mu_1. \quad (13)
\]

2. Conclusion

In this review article, the relationships among moment sequences, complete monotonic sequences, and minimal completely monotonic sequences are introduced. We also introduce the relations between completely monotonic functions and completely monotonic sequences or related sequences. Necessary conditions, sufficient conditions, or necessary and sufficient conditions for a sequence to be moment sequence, completely monotonic sequence, or minimal completely monotonic sequence are introduced.
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