Baire Category Theorem in Generalized M-fuzzy Metric Space

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Abstract:
The aim of this paper is to study some topological concepts discussed in generalized $\mathcal{M}$-fuzzy metric space.

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1. Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [12]. Since then, many authors have expansively developed the theory of fuzzy sets and applications. Especially Kaleva and Seikkala [4] Kramosil and Michalek [5] have introduced the concepts of fuzzy metric spaces in different ways. We introduced generalized $\mathcal{M}$-fuzzy metric space [7]. The main objective of this paper is to study and prove the Baire Category theorem in our Generalized $\mathcal{M}$-fuzzy metric space.

2. Preliminaries

Definition 2.1. [7]

A 3-tuple $(X, \mathcal{M}, \ast)$ is called a generalized $\mathcal{M}$-fuzzy metric space if $X$ is an arbitrary nonempty set, $\ast$ is a continuous t-norm and $\mathcal{M} : X^n \times (0, \infty) \to [0, 1]$, $n \geq 3$ satisfying the following conditions, for each $x_1, x_2, \ldots, x_n, x'_n \in X$ and $t, s > 0$.

(i) $\mathcal{M}(x_1, x_2, \ldots, x_n, t) > 0$

(ii) $\mathcal{M}(x_1, x_2, \ldots, x_n, t) = 1$ for all $t > 0$ if and only if $x_1 = x_2 = \ldots = x_n$.

(iii) $\mathcal{M}(x_1, x_2, \ldots, x_n, t) = \mathcal{M}(p(x_1, x_2, \ldots, x_n), t)$ where $p$ is a permutation function.

(iv) $\mathcal{M}(x_1, x_2, \ldots, x_n, t+s) \geq \mathcal{M}(x_1, x_2, \ldots, x_{n-1}, x'_n, t) \ast \mathcal{M}(x'_n, x_n, \ldots, x_n, s)$

(v) $\mathcal{M}(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous.

(vi) $\mathcal{M}(x_1, x_2, \ldots, x_n, t) \to 1$ as $t \to \infty$. 
Example 2.2. [7]
Let \((X, \mathcal{M}, \ast)\) be a modified fuzzy metric as in [3] which satisfy the additional condition \(M(x, y, t) \to 1\) as \(t \to \infty\). Define \(\mathcal{M}(x_1, x_2, ..., x_n, t) = M(x_1, x_2, t) \ast M(x_2, x_3, t) \ast ... \ast M(x_{n-1}, x_n, t)\) for every \(x_1, x_2, ..., x_n \in X\). Then \((X, \mathcal{M}, \ast)\) is a generalized \(\mathcal{M}\)-fuzzy metric space.

Example 2.3. [9]
Consider \(X = \mathbb{R}\). Let \(\ast\) be the product norm defined by \(a \ast b = ab\). Define \(\mathcal{M}(x_1, x_2, ..., x_n, t) = \frac{t}{t+|x_1-x_2|+...+|x_{n-1}-x_n|}\). Then \((X, \mathcal{M}, \ast)\) is a generalized \(\mathcal{M}\)-fuzzy metric space.

Definition 2.4. [7]
Let \((X, \mathcal{M}, \ast)\) be a generalized \(\mathcal{M}\)-fuzzy metric space and let \(x \in X\), \(0 < r < 1\), \(t > 0\) we define \(B_{\mathcal{M}}(x, r, t) = \{y \in X / \mathcal{M}(y, ..., y, x, t) > 1-r\}\) called open ball.

Definition 2.5. [7]
Let \((X, \mathcal{M}, \ast)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. A subset \(A\) of \(X\) is said to be open if for each \(x \in A\) there is a \(0 < r < 1\), \(t > 0\) such that \(B_{\mathcal{M}}(x, r, t) \subseteq A\).

Theorem 2.6. [7]
Let \((X, \mathcal{M}, \ast)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. Then every open ball is an open set.

Theorem 2.7. [7]
Let \((X, \mathcal{M}, \ast)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. Then \(\tau_{\mathcal{M}} = \{A / A\text{ is open in } X\}\) is a topology on \(X\) called topology induced by \(\mathcal{M}\). We call \((X, \tau_{\mathcal{M}})\) is a topological space induced by \(\mathcal{M}\).

Theorem 2.8. [7]
Every generalized \(\mathcal{M}\)-fuzzy metric space is a Hausdorff space.

3. Some Results in the topology induced by Generalized \(\mathcal{M}\)-fuzzy metric space
Here we study some topological concepts which are familiar results in topology but the tools we used to prove the results are techniques involved in generalized \(\mathcal{M}\)-fuzzy metric space.
Definition 3.1.
Let \((X, \mathcal{M}, *)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. Then \(X\) is said to be first countable space if for every \(x \in X\) has a countable local base.

Theorem 3.2.
The generalized \(\mathcal{M}\)-fuzzy metric space \((X, \mathcal{M}, *)\) is a first countable space.

Proof.
Let \(x \in X\) and \(\mathcal{B}_x = \{ B_{\mathcal{M}}(x, \frac{1}{n}, \frac{1}{n}), n = 1, 2, \ldots \} \). Then \(\mathcal{B}_x\) is a countable collection of openset in \(X\) that contains the point \(x\). If \(G\) is an openset containing \(x\), then there are \(r, t > 0, 0 < r < 1\) such that \(B_{\mathcal{M}}(x, r, t) \subset G\). Choose \(n \in \mathbb{N}\) such that \(n > \min\{ \frac{1}{r}, \frac{1}{t} \}\). Then \(B_{\mathcal{M}}(x, \frac{1}{n}, \frac{1}{n}) \subset B_{\mathcal{M}}(x, r, t) \subset G\).
Thus \(\mathcal{B}_x\) is a countable local base at \(x\). □

Definition 3.3.
Let \((X, \mathcal{M}, *)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. We define a closed ball with centre \(x \in X\) and radius \(r, 0 < r < 1, t > 0\) as \(B_{\mathcal{M}}[x, r, t] = \{ y \in X / \mathcal{M}(y, y, ..., y, x, t) \geq 1-r \}\).

Definition 3.4.
Let \((X, \mathcal{M}, *)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. Let \(A \subseteq X\). Then \(A\) is closed if it contains all its limit points.

Theorem 3.5.
Let \((X, \mathcal{M}, *)\) be a generalized \(\mathcal{M}\)-fuzzy metric space. The set \(A \subseteq X\) is open if and only if its complement is a closed set.

Proof.
Let \(y\) be a limit point of \(A^C\). Then there is a sequence \(x_n \in A^C\) such that \(x_n \to y\). Suppose \(y \not\in A^C\), then \(y \in A\), there is a \(0 < r < 1, t > 0\) such that \(B_{\mathcal{M}}(y, r, t) \subseteq A\). Since \(x_n \in A^C\) for all \(n\), \(\mathcal{M}(x_n, y, ..., y, t) \leq 1-r\). Choose \(n_0\) sufficiently large so that \(\frac{1}{n} < r\). We have \(\mathcal{M}(x_n, y, ..., y, t) \leq 1-r < 1-\frac{1}{n_0}\). i.e., \(\mathcal{M}(x_n, y, ..., y, t) < 1-\frac{1}{n_0}\), which is contradiction to \(x_n \to y\).

Suppose \(A\) is a closed set. Let \(y \in A^C\). Then \(y \not\in A\). Since \(A\) is closed, \(y\) is not a limit point of \(A\). It is enough if we find a \(0 < r < 1, t > 0\) such that \(B_{\mathcal{M}}(y, r, t) \subseteq A^C\). Suppose not, choose \(x_n \in A\).
B_M(y, \frac{1}{n}, \frac{1}{n}) and x_n \in A. Then M(x_n, y, ..., y, \frac{1}{n}) > 1 - \frac{1}{n} \forall n > 0. Therefore x_n \to y with x_n \in A, which is a contradiction to y \notin A. Hence proved. □

**Theorem 3.6.**

In a generalized \(\mathcal{M}\)-fuzzy metric space, every closed ball is a closed set.

**Proof.**

Let (X, \(\mathcal{M}\), *) be a generalized \(\mathcal{M}\)-fuzzy metric space. Consider a closed ball \(B_M(x, r, t)\). Let y be a limit point of \(B_M(x, r, t)\). Since X is First countable, there exists a sequence \(\{y_n\}\) in \(B_M(x, r, t)\) such that \(y_n \to y\). Hence by definition, \(\mathcal{M}(y_n, y, ..., y, t) \to 1\) for all \(t > 0\) and \(\mathcal{M}(x, y_n, ..., y_n, t) > 1 - r\) for all \(n\). Now for a given \(\varepsilon > 0\), we have \(\mathcal{M}(x, y, ..., y, t+\varepsilon) \geq \mathcal{M}(x, y_n, ..., y_n, t) \ast \mathcal{M}(y_n, y, ..., y, \varepsilon)\). Hence

\[
\mathcal{M}(x, y, ..., y, t+\varepsilon) \geq \lim_{n \to \infty} \mathcal{M}(x, y_n, ..., y_n, t) \ast \lim_{n \to \infty} \mathcal{M}(y_n, y, ..., y, \varepsilon)
\geq (1-r) \ast 1
= 1-r.
\]

In particular, for any \(n \in \mathbb{N}\). It is clear that \(\mathcal{M}(x, y, ..., y, t+\frac{1}{n}) \geq 1-r\). Hence

\[
\mathcal{M}(x, y, ..., y, t) = \lim_{n \to \infty} \mathcal{M}(x, y, ..., y, t+\frac{1}{n})
\geq 1-r.
\]

and so \(y \in B_M(x, r, t)\). This means that \(B_M(x, r, t)\) is a closed set.

**Definition 3.7.**

Let (X, \(\mathcal{M}\), *) be a generalized \(\mathcal{M}\)-fuzzy metric space and let A and B are disjoint closed subsets of X. X is said to be normal space, if there exists disjoint opensets U and V such that \(A \subseteq U\), \(B \subseteq V\).

**Theorem 3.8.**

Every generalized \(\mathcal{M}\)-fuzzy metric space is a normal space.

**Proof.**

Let (X, \(\mathcal{M}\), *) be the given generalized \(\mathcal{M}\)-fuzzy metric space. Let F and G be two disjoint closed sets. Let \(x \in F\). Then \(x \in G^C\). Since \(G^C\) is open, there exists \(0 < r_x < 1\), \(t_x > 0\) such that \(B_M(x, r_x, t_x) \subset G^C\) and so \(B_M(x, r_x, t_x) \cap G = \emptyset\), for all \(x \in F\). Similarly \(B_M(y, r_y, t_y) \cap F = \emptyset\), for all \(y \in G\).
Let \( s = \min\{r_x, r_y, t_x, t_y\} \). It is possible to find one \( 0 < s_0 < s \) such that \((1-s_0) * (1-s_0) > 1-s\). Define

\[
U = \bigcup_{x \in F} B_M(x, s_0, \frac{s}{2})
\]

and

\[
V = \bigcup_{y \in G} B_M(y, s_0, \frac{s}{2})
\]

Suppose \( U \cap V \neq \emptyset \), then there is a \( z \in U, z \in V \) and so \( z \in B_M(x, s_0, \frac{s}{2}) \) for some \( x \in F, y \in G \). Now,

\[
M(x, ..., x, y, s) \geq M(x, ..., x, z, \frac{s}{2}) * M(z, ..., z, y, \frac{s}{2})
\]

\[
> (1-s_0) * (1-s_0)
\]

\[
> 1-s
\]

This means that \( y \in B_M(x, r, s) \subset B_M(x, r_x, t_x) \).

It is a contradiction to \( B_M(x, r_x, t_x) \cap G = \emptyset \). Hence \( U \cap V = \emptyset \) and \( F \subseteq U, G \subseteq V \). This means that \( (X, M, *) \) is a normal space.

\[ \square \]

**Definition 3.9.**

A generalized \( M \)-fuzzy metric space \( (X, M, *) \) is said to be separable if it has a countable dense subset in \( X \).

**Theorem 3.10.**

A compact generalized \( M \)-fuzzy metric space \( (X, M, *) \) is separable.

**Proof.**

Let \( (X, M, *) \) be the given compact generalized \( M \)-fuzzy metric space. Then for given \( r, t > 0, 0 < r < 1 \), we can find \( x_1, x_2, ..., x_n \) in \( X \) such that \( X = \bigcup_{i=1}^{n} B_M(x_i, r, t) \). In particular, for each \( n \in \mathbb{N} \) we can find a finite set \( A_n = \{a_1, a_2, ..., a_n\} \) in \( X \) such that \( X = \bigcup_{i=1}^{n} (a_i, 1/n, 1/n) \). Consider \( A = \bigcup_{n=1}^{\infty} A_n \). It is clear that \( A \) is a countable dense set in \( X \).

\[ \square \]

**Theorem 3.11** (Baire Category Theorem)
Let \((X, \mathcal{M}, \ast)\) be a complete generalized \(\mathcal{M}\)-fuzzy metric space. Then countable intersection of open dense sets in \(X\) is a dense set.

**Proof.**

Let \(D_1, D_2, \ldots\) be a countable collection of dense sets in \(X\). Consider \(D = \bigcap_{n=1}^{\infty} D_n\). Let \(B\) be an open set, it is enough if we prove \(B \cap D \neq \emptyset\). Since \(B\) is open and \(D_1\) is a dense set in \(X\), \(B \cap D_1 \neq \emptyset\). Let \(x_1 \in B \cap D_1\). Since \(B\) and \(D_1\) are open sets, \(B \cap D_1\) is an open set and so there is a \(r_1 \in (0, 1), t_1 > 0\) such that \(B \mathcal{M}(x_1, r_1, t_1) \subset B \cap D_1\).

Choose a rational number \(n_1\) such that \(B \mathcal{M}(x_1, 1/n_1, 1/n_1) \subset B \mathcal{M}(x_1, 1/n_1, 1/n_1) \subset B \cap D_1\).

Consider \(B_1 = B \mathcal{M}(x_1, 1/n_1, 1/n_1)\). Since \(B_1\) is open and \(D_2\) is dense in \(X\), \(B_1 \cap D_2 \neq \emptyset\). Let \(x_2 \in B_1 \cap D_2\). Since \(B_1 \cap D_2\) is open, there is \(r_2, t_2\) such that \(B \mathcal{M}(x_2, r_2, t_2) \subset B_1 \cap D_2\).

Choose a rational number \(n_2\) such that \(n < n_2\) and
\[
B \mathcal{M}(x_2, 1/n_2, 1/n_2) \subset B \mathcal{M}(x_2, 1/n_2, 1/n_2) \subset B \mathcal{M}(x_2, r_2, t_2) \subset B_1 \cap D_2.
\]

Consider \(B_2 = B \mathcal{M}(x_2, 1/n_2, 1/n_2)\). Proceeding like this we can get a sequence \(\{x_n\}\) with \(x_n \in B \mathcal{M}(x_{n-1}, r_{n-1}/n, r_{n-1}/n)\). For any \(t > 0\), choose \(n_0\) such that \(\frac{1}{n_0} < t\). Now for any \(n, m \geq n_0\),
\[
\mathcal{M}(x_n, x_m, \ldots, x_m, t) \geq \mathcal{M}(x_n, x_m, \ldots, x_m, \frac{1}{n_0}),
\]
By the construction of \(\{x_n\}\), if \(m > n\), \(x_m \in B \mathcal{M}(x_n, \frac{1}{n}, \frac{1}{n})\) and so
\[
\mathcal{M}(x_n, x_m, \ldots, x_m, t) \geq \mathcal{M}(x_n, x_m, \ldots, x_m, \frac{1}{n_0})
\]
\[
> \mathcal{M}(x_n, x_m, \ldots, x_m, \frac{1}{n})
\]
\[
> 1 - \frac{1}{n} \rightarrow 1 \text{ as } n, m \rightarrow \infty
\]
Hence \(\{x_n\}\) is a Cauchy sequence. Since \((X, \mathcal{M}, \ast)\) is complete, there is a \(x \in X\) such that \(\{x_n\} \rightarrow x\). Since \(x_k \in B \mathcal{M}(x_n, 1/n, 1/n)\), for all \(k \geq n\). \(x \in B \mathcal{M}(x_n, 1/n, 1/n) \subset B_{n-1} \cap D_n\) for all \(n\). i.e., \(x \in D_n\) for all \(n\) and so \(x \in \bigcap_{n=1}^{\infty} D_n\). Also by our construction of \(B \mathcal{M}(x_n, 1/n, 1/n)\) it is clear that \(x \in B\).

Hence \(B \cap \left(\bigcap_{n=1}^{\infty} D_n\right) \neq \emptyset\). \(\square\)

**References**

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