RESEARCH ARTICLE OPEN ACCESS

Exact solutions of space-time fractional reaction-diffusion equations by (G'/G)-expansion method

Metin Bayrak*

*(Kocaeli University, Department of Mathematics, Kocaeli, Turkey
Email: mbayrak@kocaeli.edu.tr)

Abstract:

In this paper, the (G'/G)-expansion method for solving nonlinear fractional reaction-diffusion equations is introduced. (G'/G)-expansion method algorithm is tested on fractional Fitzhugh-Nagumo, Newell-Whitehead-Segel and Zeldovich equations. New types of exact analytical solutions are obtained with the help of symbolic computation. It is shown that the method provides a straightforward and mathematical tool for solving nonlinear fractional reaction-diffusion equations.

Keywords — **The (G'/G)-expansion method , fractional reaction-diffusion equations , modified Riemann-Liouville derivative.**

I. INTRODUCTION

In the last few years, the nonlinear fractional differential equations in mathematical physics have been considerable interest in many fields such as biophysics, blood flow phenomena, signal processing, control theory, electrical circuits and finance [1-6]. The fractional partial differential equations have been investigated by many authors [7-9]. In recently, some effective methods have proposed for determination of the exact solutions for some space and time fractional differential equations, such as the exp-function method [10-11], the fractional sub-equation method [12-15], the (G'/G)-expansion method [16-20] and the first integral method [21].

Newell-Whitehead-Segel (NWS) equation is an important nonlinear reaction-diffusion equation and usually is used to model the transmission of nerve impulse, also used in circuit theory, biology and the area of population genetics as mathematical models while the Zeldovich has been reported to arise in combustion theory. These equations arise if the Fitzhugh-Nagumo (FN) equation is reduced. The Fitzhugh-Nagumo (FN) equation [22] takes this form:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-u(1-u)(a-u) \tag{1}$$

If $a=-1$ in Eq(1), then the Newell-Whitehead-Segel(NWS) [23-26] is formed to be

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-u(1-u)^2 \tag{2}$$

and if $a=0$ the Zeldovich equation is formed. The Zeldovich equation [27] is represented as

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u^2-u^3 \tag{3}$$

This study is motivated by the need to propose the (G'/G)-expansion method to construct exact analytical solutions of fractional reaction-diffusion equations in the sense of modified Riemann-Liouville derivative by Jumarie [28]. The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)d\xi, 0<\alpha<1\\ (f^{(n)}(t))^{\alpha-n}, \quad n\le\alpha<n+1, n\ge 1\end{cases} \tag{4}$$

Some useful formulas and results of Jumarie's modified Riemann-Liouville derivative were summarized in [27], four of them are

$$D_t^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha}, \gamma>0 \tag{5}$$

$$D_t^\alpha c=0, c=constant \tag{6}$$

$$D_t^\alpha\big(cf(t)\big)=cD_t^\alpha\big(f(t)\big), c=constant \tag{7}$$

$$D_t^\alpha\big(af(t)+bg(t)\big)=aD_t^\alpha\big(f(t)\big)+aD_t^\alpha\big(g(t)\big), \quad a,b=constant \tag{8}$$

which will be used in the following sections.

The structure of this paper is as follows. In Section 2, we define the (G'/G)-expansion method for solving space-time fractional reaction diffusion equations. In Section 3-5, we applied the proposed method to the space-time fractional Fitzhugh-Nagumo, Newell-Whitehead-Segel and Zeldovich equations. Finally, we present some conclusions.

II. The (G'/G)-expansion method

In this section, we outline the main steps of the (G'/G)-expansion method.

Suppose that a fractional partial differential equation, say in two independent variables $x$and t is given by

$$P\left(u, D_t^{\alpha} u, D_x^{\beta} u, D_t^{2\alpha} u, D_x^{2\beta} u, \ldots\right) = 0, 0 < \alpha, \beta < 1 \tag{9}$$

where u is an unknown function, and P is a polynomial of u and its various partial derivatives including the highest order derivatives and nonlinear terms.

Step 1. By using the traveling wave transformation:

$$u(x,t) = U(\xi), \xi = k\frac{x^{\alpha}}{\Gamma(\alpha+1)} - c\frac{t^{\beta}}{\Gamma(\beta+1)} \tag{10}$$

where $k$and c are nonzero arbitrary constants to be determined later, Eq.(9) is reduced to the following nonlinear ordinary differential equation

$$Q(U, -cU', kU', c^2U'', k^2U'', \ldots) = 0 \tag{11}$$

where $U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{dU^2}$.

Step 2. Suppose that the solution of Eq. (11) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, a_m \neq 0 \tag{12}$$

where $a_i\ (i = 0, \pm1, \pm2, \ldots, \pm m)$ are constants, while $G(\xi)$ satisfies the following second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{13}$$

where $\lambda$and $\mu$are real constants.

Step 3. The positive integer $m$can be determined by balancing between the highest order derivatives and the nonlinear terms in Eq. (11).

Step 4. By substituting Eq.(12) into Eq.(11) and using Eq.(13) we collect all terms with the same order of (G'/G). Equating each coefficient of the resulting polynomial to zero, we yield a set of algebraic equations for $a_i\ (i = 0, \pm1, \pm2, \ldots, \pm m), \lambda, \mu, c, k$.

Step 5. By solving the equation system in Step 4 and substituting $a_i\ (i = 0, \pm1, \pm2, \ldots, \pm m), \lambda, \mu, c, k$ and the general solutions of Eq.(12) into Eq.(11), we will get more traveling wave solutions of Eq.(9).

III. Application to Fitzhugh-Nagumo (FN) Equation

Firstly, we take into account the (G'/G)-expansion method to the space-time fractional Fitzhugh-Nagumo (FN) equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(a-u), 0 < \alpha \le 1 \tag{14}$$

We present the following transformation

$$u(x,t) = U(\xi), \xi = k\frac{x^{\alpha}}{\Gamma(\alpha+1)} - c\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{15}$$

where k and c are nonzero arbitrary constants.

Substituting Eq.(15) with Eq.(4) and Eq.(14) can be turned into an ODE

$$-cU' - k^2U'' - U(1-U)(U-a) = 0 \tag{16}$$

where $U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{dU^2}$.

By using the ansatz (16), for the linear term of highest order U'' with the highest order nonlinear term U^3. By simple calculation, we have balancing U'' with U^3 in Eq.(16) gives

$$m + 2 = 3m \tag{17}$$

So that

$$m = 1. \tag{18}$$

Suppose that the solutions of Eq.(16) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), a_1 \neq 0 \tag{19}$$

By using Eq.(13) from Eq.(19) we have

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu \tag{20}$$

$$U''(\xi) = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + a_1\lambda\mu \tag{21}$$

and

$$U^2(\xi) = {a_1}^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + {a_0}^2 \tag{22}$$

and

$$U^3(\xi) = {a_1}^3\left(\frac{G'}{G}\right)^3 + 3a_0{a_1}^2\left(\frac{G'}{G}\right)^2 + 3{a_0}^2a_1\left(\frac{G'}{G}\right) + {a_0}^3 \quad (23)$$

Substituting Eq.(19)-(23) into Eq.(16), collecting the coefficients of $\left(\frac{G'}{G}\right)^i, (i = 0,1,2,3)$ and set it to zero, we obtain the system

$$\begin{aligned} -2k^2a_1+{a_1}^3 &= 0 \\ a_1c - 3\lambda k^2a_1 + 3a_0{a_1}^2 - (a+1){a_1}^2 &= 0 \\ \lambda a_1c - k^2(\lambda^2a_1 + 2a_1\mu) - (a+1)2a_0a_1 & \\ + 3{a_0}^2a_1 + aa_1 &= 0 \\ \mu a_1c - \lambda k^2a_1\mu - (a+1){a_0}^2 + {a_0}^3 + aa_0 &= 0 \end{aligned} \quad (24)$$

We can solve this system by symbolic computation get sets of solutions.

Case 1.

$$a_0 = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2 - 4\mu}}, a_1 = \pm\frac{a}{\sqrt{\lambda^2 - 4\mu}}$$

$$c = \pm\frac{a(a-2)}{2\sqrt{\lambda^2-4\mu}}, k = \pm\frac{a}{\sqrt{2}\sqrt{\lambda^2-4\mu}} \quad (25)$$

where $\lambda$and μ are arbitrary constants.

By using Eq.(25) expression (19) can be written as

$$U(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} \pm \frac{a}{\sqrt{\lambda^2-4\mu}}\left(\frac{G'}{G}\right) \quad (26)$$

where $\xi = \pm\frac{a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm a(a-2)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

Substituting the general solutions of Eq.(13) into Eq.(26), we have two types of exact solutions of the space-time fractional Fitzhugh-Nagumo (FN) equation as follows:

When $\sqrt{\lambda^2 - 4\mu} > 0$,

$$U_{1,2}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} \pm \frac{a}{\sqrt{\lambda^2-4\mu}} \times \left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}\right) \quad (27)$$

where $\xi = \pm\frac{a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm a(a-2)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$

and C_1, C_2 are arbitrary constants.

When $\sqrt{\lambda^2 - 4\mu} < 0$,

$$U_{3,4}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} \pm \frac{a}{\sqrt{\lambda^2-4\mu}} \times \left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(\frac{-C_1\sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi + C_2\cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}{C_1\cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi + C_2\sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}\right) - \lambda 2\right) \quad (28)$$

where $\xi = \pm\frac{a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm a(a-2)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$

and C_1, C_2 are arbitrary constants.

Case 2.

$$a_0 = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} + \frac{1}{2}\left(1 \pm \frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right),$$

$$a_1 = \pm\frac{1-a}{\sqrt{\lambda^2-4\mu}}, c = \pm\frac{a-1}{2\sqrt{\lambda^2-4\mu}},$$

$$k = \pm\frac{1-a}{\sqrt{2}\sqrt{\lambda^2-4\mu}} \quad (29)$$

where $\lambda$and $\mu$are arbitrary constants.

By using Eq.(26), expression (19) can be written as

$$U(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} + \frac{1}{2}\left(1 \pm \frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right) \pm \frac{1-a}{\sqrt{\lambda^2-4\mu}}\left(\frac{G'}{G}\right) \quad (30)$$

where $\xi = \pm\frac{1-a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm(a-1)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

Substituting the general solutions of Eq.(13) into Eq.(30), we have two types of exact solutions of the space-time fractional Fitzhugh-Nagumo(FN) equation as follows:

When $\sqrt{\lambda^2 - 4\mu} > 0$,

$$U_{1,2}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}} + \frac{1}{2}\left(1 \pm \frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right) \pm \frac{1-a}{\sqrt{\lambda^2-4\mu}} \times \left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}\right) \quad (31)$$

where $\xi = \pm\frac{1-a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm(a-1)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$

and C_1, C_2 are arbitrary constants.

When $\sqrt{\lambda^2 - 4\mu} < 0$,

$$U_{3,4}(\xi)=\frac{a}{2}\pm\frac{a\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}+\frac{1}{2}\left(1\pm\frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right)\pm\frac{1-a}{\sqrt{\lambda^2-4\mu}}\times\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(\frac{-C_1 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}{C_1 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}\right)-\lambda 2\right) \quad (32)$$

where $\xi=\pm\frac{1-a}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)}-\frac{\pm(a-1)}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$ and C_1, C_2 are arbitrary constants.

IV. Application to Newell-Whitehead-Segel (NWS) Equation

Secondly, we apply the (G'/G)-expansion method to the space-time fractional NWS

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}+u(1-u)^2, 0<\alpha\le 1 \quad (33)$$

Using traveling wave variable

$$u(x,t)=U(\xi), \xi=k\frac{x^\alpha}{\Gamma(\alpha+1)}-c\frac{t^\alpha}{\Gamma(\alpha+1)} \quad (34)$$

where k and c are nonzero arbitrary constants. When we substitute Eq.(34) with Eq.(5) and Eq.(33) can be turned into an ODE

$$-cU'-k^2U''-U(1-U)^2=0 \quad (35)$$

where $U'=\frac{dU}{d\xi}, U''=\frac{d^2U}{dU^2}$.

By using the ansatz (35), for the linear term of highest order U'' with the highest order nonlinear term U^3, balancing U'' with U^2 in Eq.(35) gives

$$m+2=3m \quad (36)$$

So that

$$m=1. \quad (37)$$

Suppose that the solutions of Eq.(35) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right), a_1\neq 0 \quad (38)$$

By using Eq.(13) from Eq.(38) we have

$$U'(\xi)=-a_1\left(\frac{G'}{G}\right)^2-a_1\lambda\left(\frac{G'}{G}\right)-a_1\mu \quad (39)$$

$$U''(\xi)=2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2+(2a_1\mu+a_1\lambda^2)\left(\frac{G'}{G}\right)+a_1\lambda\mu \quad (40)$$

and

$$U^3(\xi)=a_1{}^3\left(\frac{G'}{G}\right)^3+3a_0a_1{}^2\left(\frac{G'}{G}\right)^2+3a_0{}^2a_1\left(\frac{G'}{G}\right)+a_0{}^3 \quad (41)$$

Substituting Eq.(38)-(41) into Eq.(35), collecting the coefficients of $\left(\frac{G'}{G}\right)^i, (i=0,1,2,3)$ and set it to zero, we obtain the system

$$-2k^2a_1+a_1{}^3=0$$
$$a_1c-3\lambda k^2a_1+3a_0a_1{}^2=0$$
$$\lambda a_1c-k^2(\lambda^2a_1+2a_1\mu)-a_1+3a_0{}^2a_1=0$$
$$\mu a_1c-\lambda k^2a_1\mu+a_0{}^3-a_0=0 \quad (42)$$

Solving this system by symbolic computation gives

$$a_0=\pm\frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}, a_1=\pm\frac{1}{\sqrt{\lambda^2-4\mu}}$$
$$c=\pm\frac{3}{2\sqrt{\lambda^2-4\mu}}, k=\pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}} \quad (43)$$

where λ and μ are arbitrary constants.

By using Eq.(43) expression (38) can be written as

$$U(\xi)=\pm\frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\left(\frac{G'}{G}\right) \quad (44)$$

where $\xi=\pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)}-\frac{\pm 3}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

By substituting the general solutions of Eq.(13) into Eq.(44), we have two types of exact solutions of the space-time fractional NWS equation as follows:

When $\sqrt{\lambda^2-4\mu}>0$,

$$U_{1,2}(\xi)=\pm\frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\times\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)-\frac{\lambda}{2}\right) \quad (45)$$

where $\xi=\pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)}-\frac{\pm 3}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

and C_1, C_2 are arbitrary constants.

When $\sqrt{\lambda^2-4\mu}<0$,

$$U_{3,4}(\xi) = \pm\frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\times\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(\frac{-C_1 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}{C_1 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}\right)-\lambda 2\right) \quad (46)$$

where $\xi = \pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm 3}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$ and C_1, C_2 are arbitrary constants.

V. Application to zeldovich Equation

Thirdly, we consider the (G'/G)-expansion method to the space-time fractional Zeldovich equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + u^2 - u^3, 0 < \alpha \le 1 \quad (47)$$

Using traveling wave variable

$$u(x,t) = U(\xi), \xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} \quad (48)$$

where k and c are nonzero arbitrary constants.

When we substitute Eq.(48) with Eq.(5) and Eq.(47) can be reduced into an ODE

$$-cU' - k^2U'' - U^2 + U^3 = 0 \quad (49)$$

where $U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{dU^2}$.

By the same procedure as illustrated in the Section 3, we can determine value of m by balancing U^3 and U'' in Eq. (49). We find $m = 1$. We can suppose that the solutions of Eq.(47) is of the form:

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), a_1 \ne 0 \quad (50)$$

By using Eq.(50), from Eq(13) we have

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu \quad (51)$$

$$U''(\xi) = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + a_1\lambda\mu \quad (52)$$

$$U^2(\xi) = {a_1}^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + {a_0}^2 \quad (53)$$

and

$$U^3(\xi) = {a_1}^3\left(\frac{G'}{G}\right)^3 + 3a_0{a_1}^2\left(\frac{G'}{G}\right)^2 + 3{a_0}^2a_1\left(\frac{G'}{G}\right) + {a_0}^3 \quad (54)$$

Substituting Eq.(50)-(54) into Eq.(47), collecting the coefficients of $\left(\frac{G'}{G}\right)^i, (i = 0,1,2,3)$ and set it to zero, we obtain the system

$$\begin{aligned} -2k^2a_1 + {a_1}^3 &= 0 \\ a_1c - 3\lambda k^2a_1 + 3a_0{a_1}^2 - {a_1}^2 &= 0 \\ \lambda a_1c - k^2(\lambda^2a_1 + 2a_1\mu) - 2a_0a_1 + 3{a_0}^2a_1 &= 0 \\ \mu a_1c - \lambda k^2a_1\mu - {a_0}^2 + {a_0}^3 &= 0 \end{aligned} \quad (55)$$

Solving this system by simple calculation gives

$$a_0 = \frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}, a_1 = \pm\frac{1}{\sqrt{\lambda^2-4\mu}}$$

$$c = \pm\frac{1}{2\sqrt{\lambda^2-4\mu}}, k = \pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}} \quad (56)$$

where λ and μ are arbitrary constants.

By using Eq.(56) expression (50) can be written as

$$U(\xi) = \frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\left(\frac{G'}{G}\right) \quad (57)$$

where $\xi = \pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm 1}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

Substituting the general solutions of Eq.(13) into Eq.(57), we have two types of exact solutions of the space-time fractional Zeldovich equation as follows:

$$U_{1,2}(\xi) = \frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\times\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)-\frac{\lambda}{2}\right) \quad (58)$$

where $\xi = \pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm 1}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.

and C_1, C_2 are arbitrary constants.

When $\sqrt{\lambda^2-4\mu} < 0$,

$$U_{3,4}(\xi) = \frac{1}{2}\pm\frac{\lambda}{2}\frac{1}{\sqrt{\lambda^2-4\mu}}\pm\frac{1}{\sqrt{\lambda^2-4\mu}}\times\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(\frac{-C_1 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}{C_1 cos\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi+C_2 sin\frac{\sqrt{-\lambda^2+4\mu}}{2}\xi}\right)-\lambda 2\right) \quad (59)$$

where $\xi = \pm\frac{1}{\sqrt{2}\sqrt{\lambda^2-4\mu}}\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\pm 1}{2\sqrt{\lambda^2-4\mu}}\frac{t^\alpha}{\Gamma(\alpha+1)}$.
and C_1, C_2 are arbitrary constants.

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam Elsevier, 2006.

[2] I. Podlubny, *Fractional Differential Equation* , CA:Academic Press, San Diego, 1999.

[3] J. Sabatier, O.P. Agarwal and JAT Machado (eds.) Advances in fractional calculus: theoretical developments and applications in physics and engineering, Dordrecht: Springer, 2007.

[4] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Amsterdam: Gordon and Breach, 1993.

[5] K. S. Miller and B. Ross, An introduction to the fractional calculus and Fractional differential equations, Wiley: New York, 1993.

[6] R. Fitzhugh, Impulse and physiological states in models of nerve membrane, *Biophys. J.* vol. 1, pp. 445-466, 1961.

[7] K. A. Gepreel and S. Omran, The exact solutions for the nonlinear partial fractional differential equations, *Chin. Phys. B* vol.21, pp.110-204, 2012.

[8] K. A. Gepreel, T. A. Nofal and A. A. Al-Thobaiti, Numerical solutions for the nonlinear partial fractional Zakharov-Kuznetsov equations with time and space fractional, *Sci. Res. Essays,* vol.9, pp.471-482,2014.

[9] K. A. Gepreel, Explicit Jacobi elliptic exact solutions for nonlinear partial fractional differential equations, *Adv. Differ. Equ.*,vol.2014,pp. 286-300, 2014 .

[10] A. Bekir, O. Guner and A. C. Cevikel, Fractional complex transform and exp-function methods for fractional differential equations, *Abstr. Appl. Anal.*,vol. 2013 ,pp. 426- 462,2013.

[11] B. Zheng, Exp-function method for solving fractional partial differential equations, *Sci. World* ., vol.2013,pp. 465-723,2013.

[12] K. A. Gepreel and A. A. Al-Thobaiti, Exact solution of nonlinear partial differential equations using the fractional sub-equation method, *Indian J. Phys.*,vol. 88, pp. 293 - 300,2014.

[13] S. Guo, L. Mei, Y. Li and Y. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A* ,vol. 376,pp.407 -411,2012.

[14] B. Tong, Y. He, L. Wei and X. Zhang, A generalized fractional sub-equation method for fractional differential equations with variable coefficients, *Phys. Lett. A*,vol. 376 ,pp.2588 -2590,2012.

[15] S. Zhang and H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A* ,vol. 375,pp. 1069- 1073, 2011.

[16] A. Bekir and O. Guner, Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method, *Chin. Phys. B*, vol. 22 ,pp. 110-202,2013.

[17] B. Zheng, (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics, *Commun. Theor. Phys.* ,vol. 58,pp. 623 -630,2012.

[18] M. Bayrak and M.A. Bayrak, Exact solutions of some transonic gas flows equations, *European International Journal of Science and Technology* ,vol. 7 ,pp. 15—20,2018.

[19] M.A. Bayrak and M. Bayrak, Analytical solution of time-fractional nonlinear Benjamin-Bona-Mahony equation by residual power series method, *International journal of Applied mathematics*, vol. 31,pp. 251—262, 2018.

[20] M.A. Bayrak, Application of the (G'/G)-expansion method for some space-time fractional partial differential equations, *Communications Series A1:Mathematics and Statics*, vol. 67 ,pp. 60 -67, 2018.

[21] B. Lu, The first integral method for some time fractional differential equations, *J. Math. Anal. Appl*. ,vol. 395 ,pp. 684 -693, 2012.

[22] M. Nadjafikhah and A. Mahdavi, Approximate symmetry analysis of a class of perturbed nonlinear reaction-diffusion equations, *Abstr. Appl.*reaction-diffusion equations, *Abstr. Appl. Anal.* ,vol. 2013 ARTICLE ID 395847, 7 pages, DOI:10.1155/2013/395847 2013.

[23] B. A. Malomed, Stability and Grain boundaries in the dispersive Newell-Whitehead-Segel equation, *Phys. Scripta*,vol. 57 ,pp. 115-119,1998.

[24] W. K. Zahra and M.S. El-Azab, Cubic B-Spline collocation algorithm for the numerical solution of Newell-Whitehead-Segel type equations, *Elect. J. Math. Anal.Appl*, vol. 2 ,pp. 81-100, 2014.

[25] A. Saravanan and N. Magesh, A comparison between the Reduced Differential Transform method and the Adomian Decomposition method for the Newell-Whitehead-Segel equation, *J. Egypt. Math. Soc*, vol. 21 ,pp. 259-265,2013.

[26] P. Pue-on, Laplace Adomian Decomposition method for solving Newell-Whitehead-Segel equation *Appl. Math. Sci,*vol. 7, pp.6593-6600,2013.

[27] R. Jiwari, R. K. Gupta, and V. Kumar, Polynomial differential quadrature method for numerical solutions of the generalized Fitzhugh–Nagumo equation with time dependent coefficients, *Ain Shams Eng. J.,*vol. 5,pp. 1343-1350, 2013.

[28] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, *Comput. Math. Appl.*,vol. 1 ,pp. 1367—1376, 2006.