Exact solutions of space-time fractional reaction-diffusion equations by (G'/G)-expansion method

Metin Bayrak*
*(Kocaeli University, Department of Mathematics, Kocaeli, Turkey
Email: mbayrak@kocaeli.edu.tr)

Abstract:
In this paper, the (G'/G)-expansion method for solving nonlinear fractional reaction-diffusion equations is introduced. (G'/G)-expansion method algorithm is tested on fractional Fitzhugh-Nagumo, Newell-Whitehead-Segel and Zeldovich equations. New types of exact analytical solutions are obtained with the help of symbolic computation. It is shown that the method provides a straightforward and mathematical tool for solving nonlinear fractional reaction-diffusion equations.

Keywords — The (G'/G)-expansion method, fractional reaction-diffusion equations, modified Riemann-Liouville derivative.

I. INTRODUCTION

In the last few years, the nonlinear fractional differential equations in mathematical physics have been considerable interest in many fields such as biophysics, blood flow phenomena, signal processing, control theory, electrical circuits and finance [1-6]. The fractional partial differential equations have been investigated by many authors [7-9]. In recently, some effective methods have proposed for determination of the exact solutions for some space and time fractional differential equations, such as the exp-function method [10-11], the fractional sub-equation method [12-15], the (G'/G)-expansion method [16-20] and the first integral method [21].

Newell-Whitehead-Segel (NWS) equation is an important nonlinear reaction-diffusion equation and usually is used to model the transmission of nerve impulse, also used in circuit theory, biology and the area of population genetics as mathematical models while the Zeldovich has been reported to arise in combustion theory. These equations arise if the Fitzhugh-Nagumo (FN) equation is reduced. The Fitzhugh-Nagumo (FN) equation [22] takes this form:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(a-u)
\]  

If \(a = -1\) in Eq(1), then the Newell-Whitehead-Segel (NWS) [23-26] is formed to be

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)^2
\]  

and if \(a = 0\) the Zeldovich equation is formed.

The Zeldovich equation [27] is represented as

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - u^3
\]  

This study is motivated by the need to propose the (G'/G)-expansion method to construct exact analytical solutions of fractional reaction-diffusion equations in the sense of modified Riemann-Liouville derivative by Jumarie [28]. The Jumarie's modified Riemann-Liouville derivative of order \(\alpha\) is defined by the following expression:

\[
D_\alpha^\alpha f(t) = \left\{ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t - \xi)^{-\alpha} (f(\xi) - f(0)) d\xi, 0 < \alpha < 1 \right\}
\]

\[
+ (f^{(n)}(\xi))^{\alpha - n}, \ n \leq \alpha < n + 1, n \geq 1
\]  

Some useful formulas and results of Jumarie's modified Riemann-Liouville derivative were summarized in [27], four of them are

\[
D_\alpha^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \ \gamma > 0
\]  

\[
D_\alpha^\alpha c = 0, c = constant
\]  

\[
D_\alpha^\alpha (cf(t)) = cD_\alpha^\alpha (f(t)), c = constant
\]  

\[
D_\alpha^\alpha (af(t) + bg(t)) = aD_\alpha^\alpha (f(t)) + aD_\alpha^\alpha (g(t)), a, b = constant
\]
which will be used in the following sections.

The structure of this paper is as follows. In Section 2, we define the (G'/G)-expansion method for solving space-time fractional reaction diffusion equations. In Section 3-5, we applied the proposed method to the space-time fractional Fitzhugh-Nagumo, Newell-Whitehead-Segel and Zeldovich equations. Finally, we present some conclusions.

II. THE (G'/G)-EXPANSION METHOD

In this section, we outline the main steps of the (G'/G)-expansion method.

Suppose that a fractional partial differential equation, say in two independent variables \( x \) and \( t \) is given by

\[
P\left(u, D_t^\alpha u, D_x^\beta u, D_t^2 D_x^\alpha u, D_x^2 D_t^\beta u, \ldots\right) = 0, \quad 0 < \alpha, \beta < 1
\]

where \( u \) is an unknown function, and \( P \) is a polynomial of \( u \) and its various partial derivatives including the highest order derivatives and nonlinear terms.

**Step 1.** By using the traveling wave transformation:

\[
u(x, t) = U(\xi), \quad \xi = k \frac{x^\alpha}{\Gamma(\alpha + 1)} - c \frac{t^\beta}{\Gamma(\beta + 1)}
\]

where \( k \) and \( c \) are nonzero arbitrary constants to be determined later. Eq.(9) is reduced to the following nonlinear ordinary differential equation

\[
Q(U, -cU', kU', c^2U'', k^2U'''', \ldots) = 0
\]

where \( U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{d\xi^2} \).

**Step 2.** Suppose that the solution of Eq. (11) can be expressed by a polynomial in \( G/G \) as follows:

\[
U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, a_m \neq 0
\]

where \( a_i \) are constants, while \( G(\xi) \) satisfies the following second order linear ordinary differential equation

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0
\]

where \( \lambda \) and \( \mu \) are real constants.

**Step 3.** The positive integer \( m \) can be determined by balancing between the highest order derivatives and the nonlinear terms in Eq. (11).

**Step 4.** By substituting Eq.(12) into Eq.(11) and using Eq.(13) we collect all terms with the same order of \( G/G \). Equating each coefficient of the resulting polynomial to zero, we yield a set of algebraic equations for \( a_i \) \((i = 0, \pm 1, \pm 2, \ldots, \pm m)\), \( \lambda, \mu, c, k \).

**Step 5.** By solving the equation system in Step 4 and substituting \( a_i \) \((i = 0, \pm 1, \pm 2, \ldots, \pm m)\), \( \lambda, \mu, c, k \) and the general solutions of Eq.(12) into Eq.(11), we will get more traveling wave solutions of Eq.(9).

III. APPLICATION TO FITZHUGH-NAGUMO (FN) EQUATION

Firstly, we take into account the \( (G'/G) \)-expansion method to the space-time fractional Fitzhugh-Nagumo (FN) equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - u(1 - u)(a - u), \quad 0 < \alpha \leq 1
\]

where \( a \) and \( c \) are nonzero arbitrary constants.

We present the following transformation

\[
u(x, t) = U(\xi), \quad \xi = k \frac{x^\alpha}{\Gamma(\alpha + 1)} - c \frac{t^\beta}{\Gamma(\beta + 1)}
\]

where \( k \) and \( c \) are nonzero arbitrary constants.

Substituting Eq.(15) with Eq.(4) and Eq.(14) can be turned into an ODE

\[-cU'' - k^2U'''' - U(1 - U)(U - a) = 0 \quad \text{(16)}
\]

where \( U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{d\xi^2} \).

By using the ansatz (16), for the linear term of highest order \( U'' \) with the highest order nonlinear term \( U^3 \). By simple calculation, we have balancing \( U'' \) with \( U^3 \) in Eq.(16) gives

\[
m + 2 = 3m
\]

So that

\[
m = 1.
\]

Suppose that the solutions of Eq.(16) can be expressed by a polynomial in \( \left(\frac{G'}{G}\right) \) as follows:

\[
U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0
\]

By using Eq.(13) from Eq.(19) we have

\[
U''(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu
\]

\[
U''(\xi) = 2a_1 \left(\frac{G'}{G}\right)^3 + 3a_1 \lambda \left(\frac{G'}{G}\right)^2 + (2a_1 \mu + a_1 \lambda^2) \left(\frac{G'}{G}\right) + a_1 \lambda \mu
\]

\[
U''(\xi) = a_1^2 \left(\frac{G'}{G}\right)^2 + 2a_0 a_1 \left(\frac{G'}{G}\right) + a_0^2
\]
\[ U^3(\xi) = a_1^3 \left( \frac{c'}{c} \right)^3 + 3a_0a_1^2 \left( \frac{c'}{c} \right)^2 + 3a_0^2a_1 \left( \frac{c'}{c} \right) + a_0^3 \]  

(23)

Substituting Eq.(19)-(23) into Eq.(16), collecting the coefficients of \( \left( \frac{c'}{c} \right)^i \), \( i = 0,1,2,3 \) and set it to zero, we obtain the system

\[ -2k^2a_1 + a_1^3 = 0 \]
\[ a_1c - 3\lambda k^2a_1 + 3a_0a_1^2 - (a + 1)a_1^2 = 0 \]
\[ \lambda a_1 - k^2(\lambda^2a_1 + 2a_1\mu) - (a + 1)2a_1a_0 + 3a_0^2a_1 + aa_1 = 0 \]
\[ \mu a_1 - \lambda k^2a_1 - (a + 1)a_0^2 + a_0^3 + aa_0 = 0 \]

(24)

We can solve this system by symbolic computation get sets of solutions.

**Case 1.**

\[ a_0 = \frac{a}{2} \pm \frac{a\lambda}{2\sqrt{\lambda^2 - 4\mu}}, a_1 = \pm \frac{a}{\sqrt{\lambda^2 - 4\mu}} \]
\[ c = \pm \frac{a(a-2)}{2\sqrt{\lambda^2 - 4\mu}} k = \pm \frac{a}{2\sqrt{2\lambda^2 - 4\mu}} \]

(25)

where \( \lambda \) and \( \mu \) are arbitrary constants.

By using Eq.(25) expression (19) can be written as

\[ U(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2\sqrt{\lambda^2 - 4\mu}} \pm \frac{a}{2\sqrt{\lambda^2 - 4\mu}} \left( \frac{c'}{c} \right) \]

(26)

where \( \xi = \pm \frac{a}{2\sqrt{2\lambda^2 - 4\mu}} x^a - \frac{a(a-2)}{2\sqrt{2\lambda^2 - 4\mu}} t^a \)

Substituting the general solutions of Eq.(13) into Eq.(26), we have two types of exact solutions of the space-time fractional Fitzhugh-Nagumo (FN) equation as follows:

When \( \sqrt{\lambda^2 - 4\mu} > 0 \),

\[ U_{1,2}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2\sqrt{\lambda^2 - 4\mu}} \pm \frac{a}{\sqrt{\lambda^2 - 4\mu}} \times \]
\[ \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( c_1 \cosh \frac{\lambda^2 - 4\mu}{2} \xi + c_2 \sinh \frac{\lambda^2 - 4\mu}{2} \xi \right) - \frac{\lambda}{2} \right) \]

(27)

where \( \xi = \pm \frac{a}{\sqrt{2\lambda^2 - 4\mu}} x^a - \frac{a(a-2)}{2\sqrt{2\lambda^2 - 4\mu}} t^a \)

and \( C_1, C_2 \) are arbitrary constants.

When \( \sqrt{\lambda^2 - 4\mu} < 0 \),

\[ U_{3,4}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2\sqrt{\lambda^2 - 4\mu}} \pm \frac{a}{\sqrt{\lambda^2 - 4\mu}} \times \]
\[ \left( \frac{\sqrt{\lambda^2 + 4\mu}}{2} \left( -c_1 \sinh \frac{\lambda^2 + 4\mu}{2} \xi + c_2 \cosh \frac{\lambda^2 + 4\mu}{2} \xi \right) - \frac{\lambda}{2} \right) \]

(28)

where \( \xi = \pm \frac{a}{\sqrt{2\lambda^2 + 4\mu}} x^a - \frac{a(a-2)}{2\sqrt{2\lambda^2 + 4\mu}} t^a \)

and \( C_1, C_2 \) are arbitrary constants.
\[ U_{34}(\xi) = \frac{a}{2} \pm \frac{a\lambda}{2} \frac{1}{\sqrt{\lambda^2 - 4\mu}} + \frac{1}{2} \left( 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \]

\[ \pm \frac{1 - a}{\sqrt{\lambda^2 - 4\mu}} \times \left( \frac{\sqrt{-\lambda^2 + 4\mu}}{2} c_1 \sin^2 \frac{-\lambda^2 + 4\mu}{2} \xi + c_2 \cos^2 \frac{-\lambda^2 + 4\mu}{2} \xi \right) - \lambda z \]

(32)

where \( \xi = \pm \frac{1 - a}{\sqrt{2} \sqrt{\lambda^2 - 4\mu}} \frac{x^a}{\Gamma(\alpha + 1)} \) and \( C_1, C_2 \) are arbitrary constants.

IV. APPLICATION TO NEWELL-WHITEHEAD-SEGEL (NWS) EQUATION

Secondly, we apply the (G'/G)-expansion method to the space-time fractional NWS

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1 - u)^2, \quad 0 < \alpha \leq 1 \] (33)

Using traveling wave variable

\[ u(x, t) = U(\xi), \quad \xi = k \frac{x^a}{\Gamma(\alpha + 1)} - c \frac{t^a}{\Gamma(\alpha + 1)} \] (34)

where \( k \) and \( c \) are nonzero arbitrary constants.

When we substitute Eq.(34) with Eq.(5) and Eq.(33) can be turned into an ODE

\[ -cU' - k^2 U'' - U(1 - U)^2 = 0 \] (35)

where \( U' = \frac{du}{d\xi}, U'' = \frac{d^2 u}{d\xi^2} \).

By using the ansatz (35), for the linear term of highest order \( U'' \) with the highest order nonlinear term \( U^3 \), balancing \( U'' \) with \( U^2 \) in Eq.(35) gives

\[ m + 2 = 3m \]

So that

\[ m = 1. \] (37)

Suppose that the solutions of Eq.(35) can be expressed by a polynomial in \( \frac{G'}{G} \) as follows:

\[ U(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right)^{3}, \quad a_1 \neq 0 \] (38)

By using Eq.(13) from Eq.(38) we have

\[ U'(\xi) = -a_1 \left( \frac{G'}{G} \right)^{2} - a_1 \lambda \left( \frac{G'}{G} \right) - a_1 \mu \] (39)

\[ U''(\xi) = 2a_1 \left( \frac{G'}{G} \right)^{3} + 3a_1 \lambda \left( \frac{G'}{G} \right)^{2} + (2a_1 \mu + a_1 \lambda^2) \left( \frac{G'}{G} \right) + a_1 \lambda \mu \] (40)

and

\[ U^3(\xi) = a_1^3 \left( \frac{G'}{G} \right)^{3} + 3a_0 a_1^2 \left( \frac{G'}{G} \right)^{2} + 3a_0^2 a_1 \left( \frac{G'}{G} \right) + a_0^3 \] (41)

Substituting Eq.(38)-(41) into Eq.(35), collecting the coefficients of \( \left( \frac{G'}{G} \right)^i \), \( i = 0, 1, 2, 3 \) and set it to zero, we obtain the system

\[ -2k^2 a_1 + a_1^3 = 0 \]

\[ a_1 c - 3\lambda k^2 a_1 + 3a_0 a_1^2 = 0 \]

\[ \lambda a_1 c - k^2 (\lambda^2 a_1 + 2a_1 \mu) - a_1 + 3a_0 a_1^2 = 0 \]

\[ \mu a_1 c - k^2 a_1 \mu + a_0^3 - a_0 = 0 \] (42)

Solving this system by symbolic computation gives

\[ a_0 = \pm \frac{1}{2} \pm \frac{\lambda}{2} \frac{1}{\sqrt{\lambda^2 - 4\mu}}, \quad a_1 = \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}} \]

\[ c = \pm \frac{3}{2\sqrt{\lambda^2 - 4\mu}}, \quad k = \pm \frac{1}{\sqrt{2\sqrt{\lambda^2 - 4\mu}}} \] (43)

where \( \lambda \) and \( \mu \) are arbitrary constants.

By using Eq.(43) expression (38) can be written as

\[ U(\xi) = \pm \frac{1}{2} \pm \frac{\lambda}{2} \frac{1}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}} \left( \frac{G'}{G} \right)^{3} \] (44)

where \( \xi = \pm \frac{1}{\sqrt{2\sqrt{\lambda^2 - 4\mu}} \Gamma(\alpha + 1)} - \frac{1}{\sqrt{2\sqrt{\lambda^2 - 4\mu}} \Gamma(\alpha + 1)} \).

By substituting the general solutions of Eq.(13) into Eq.(44), we have two types of exact solutions of the space-time fractional NWS equation as follows:

When \( \sqrt{\lambda^2 - 4\mu} > 0 \),

\[ U_{1,2}(\xi) = \pm \frac{1}{2} \pm \frac{1}{2} \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}} \times \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( c_1 \cosh \frac{\lambda^2 - 4\mu}{2} \xi + c_2 \sinh \frac{\lambda^2 - 4\mu}{2} \xi \right) - \frac{\lambda}{2} \right) \] (45)

where \( \xi = \pm \frac{1}{\sqrt{2\sqrt{\lambda^2 - 4\mu} \Gamma(\alpha + 1)} - \frac{1}{\sqrt{2\sqrt{\lambda^2 - 4\mu} \Gamma(\alpha + 1)}} \).

and \( C_1, C_2 \) are arbitrary constants.

When \( \sqrt{\lambda^2 - 4\mu} < 0 \),
By using Eq.(50), from Eq(13) we have

\[
U(\xi) = a_0 + a_1 \left( \frac{c'}{c} \right)^2 + 2a_0 a_1 \left( \frac{c'}{c} \right)^2 + a_0^2
\]  

Substituting Eq.(50)-(54) into Eq.(47), collecting the coefficients of \( \left( \frac{c'}{c} \right)^i, (i = 0,1,2,3) \) and set it to zero, we obtain the system

\[
a_1 c - 3\lambda k^2 a_1 + 3a_0 a_1^2 - a_1^2 = 0
\]

\[
\mu a_1 c - k^2 (\lambda^2 a_1 + 2a_1 \mu) - 2a_0 a_1 + 3a_0^2 a_1 = 0
\]

Solving this system by simple calculation gives

\[
a_0 = \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu}, a_1 = \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu}
\]

where \( \lambda \) and \( \mu \) are arbitrary constants.

By using Eq.(56) expression (50) can be written as

\[
U(\xi) = \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} + \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \left( \frac{c'}{c} \right)
\]

Substituting the general solutions of Eq.(13) into Eq.(57), we have two types of exact solutions of the space-time fractional Zeldovich equation as follows:

\[
U_{1,2}(\xi) = \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu}
\]

\[
\left( \frac{\lambda^2 - 4\mu}{2} \left( \frac{c'}{c} \right)^2 + a_1 \lambda \left( \frac{c'}{c} \right)^2 \right)
\]

where \( \xi = \frac{1}{2} \sqrt{\frac{\lambda^2 - 4\mu}{2} \frac{c'}{c} + \frac{1}{2} \sqrt{\lambda^2 - 4\mu}} \)

and \( C_1, C_2 \) are arbitrary constants.

When \( \sqrt{\lambda^2 - 4\mu} < 0 \),

\[
U_{3,4}(\xi) = \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu}
\]

\[
\left( \frac{\lambda^2 - 4\mu}{2} \left( \frac{c'}{c} \right)^2 + a_1 \lambda \left( \frac{c'}{c} \right)^2 \right)
\]

where \( \xi = \frac{1}{2} \sqrt{\frac{\lambda^2 - 4\mu}{2} \frac{c'}{c} - \frac{1}{2} \sqrt{\lambda^2 - 4\mu}} \)

and \( C_1, C_2 \) are arbitrary constants.

When \( \lambda^2 - 4\mu < 0 \),

\[
U_{3,4}(\xi) = \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu}
\]

\[
\left( \frac{\lambda^2 - 4\mu}{2} \left( \frac{c'}{c} \right)^2 + a_1 \lambda \left( \frac{c'}{c} \right)^2 \right)
\]
where \( \xi = \pm \frac{1}{\sqrt{2(1/2 - 4\mu)}} \frac{x^a}{\Gamma(\alpha + 1)} \) 
and \( C_1, C_2 \) are arbitrary constants.

REFERENCES


