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The Study of Fuzzy Differential Equations Expanding as New Branch of Fuzzy Mathematics

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Abstract:

Partial differential equations are used for modeling various physical phenomena. Unfortunately, many problems are dynamical and too complicated, developing an accurate differential equation model for such problems require complex and time consuming algorithms hardly implementable in Leveque R. J (2005). For a long time, scientist goal was to develop constructive and effective methods that reliably compute the partial differential equation with more accuracy as possible. In classical mathematics, various kinds of transforms (Fourier, Laplace, integral, wavelet) are used as powerful methods for construction of approximation models and for solution of differential or integral-differential equations Perfilieva I. (2004).

Fuzzy set theory is composed of an organized body of mathematical tools particularly well-suited for handling incomplete information, the un-sharpness of classes of objects or situations, or the gradualness of preference profiles, in a flexible way. It offers a unifying framework for modeling various types of information ranging from precise numerical, interval-valued data, to symbolic and linguistic knowledge, with a stress on semantics rather than syntax. Zimmermann H. J (2001)

Achieving high levels of precision is a very important subject in all science fields, getting a satisfied precision depending basically on the way we deal with elements in the problem Universe. For many years we were depended on crisp set theory "classical set theory" to deal with elements and sets which belong to the problem Universe, but in real world there are many application problems which can't be described nor handled by the crisp set theory Leondes C. (1998).The study of fuzzy differential equations is rapidly expanding as a new branch of fuzzy mathematics. Both theory and applications have been actively discussed over the last few years. According to Vorobiev and Seikkala (1986), the term ‘fuzzy differential equation’ was first coined in 1978. Since then, it has been a subject of interest among scientists and engineers. In the literature, the study of fuzzy differential equations has several interpretations. The first one is based on the notion of Hukuhara derivative (R. Goetschel (1986 et al)). Under this interpretation, the existence and uniqueness of the solution of fuzzy differential equations have been extensively studied by (S. Song,(2000), S. Seikkala, (1987))The concept of Hukuhara derivative was further explored by Kaleva (1987) and Seikkala (1986). Subsequently, the theory of fuzzy differential equations has been developed and fuzzy initial value problems have been studied. However, this approach produces many solutions that have an increasing length of support as the independent variable increases (T.G. Bhaskar,et al (2012)). Moreover, different formulations of the same fuzzy differential equation might lead to different solutions.

According to Diamond (2000) the approach based on the Hukuhara derivative does not produce the variety of behaviours as in the case of ordinary differential equations. This short coming has been alleviated by Hullermeier (1997), who studied a fuzzy differential equation as a family of differential inclusions. According to Bede et al. (2006) the main Short coming of Hullermeier's approach is that it does not include a ''fuzzification'' of the differential operator. The same authors also claim that the solution of a fuzzy differential equation is not necessarily a fuzzy interval-valued function. B. Bede (2007) et al is shown that in some situations, the approach based on Hullermeier's interpretation also yields different solutions.The third interpretation was suggested by Buckley and Feuring (2000) who applied the extension principle to the crisp solution of ordinary differential equations in order to obtain a solution in the fuzzy setting.In this case, different formulations of the same ordinary differential equation lead to the same solution ensuring its uniqueness. In 2005, Bede and Gal introduced a new concept of fuzzy derivatives called the generalised differentiability of fuzzy interval valued functions. In this setting, the solution of a fuzzy differential equation may have a decreasing length of support as the independent variable increases. However, it depends on the selection of the fuzzy derivatives. Moreover, different formulations of the same fuzzy differential equation will lead to different solutions as well. Therefore, the uniqueness is not ensured. In 2013 Khastan et al used the generalized differentiability to study the existence of solutions of a class of first-order linear fuzzy differential equations with periodic boundary conditions.Recently, Gasilov et al. (2011) proposed a new method to solve a system of linear differential equations with real coefficients and with an initial condition described by a vector of fuzzy intervals. The proposed method is based on properties of linear transformations. However, the authors considered a fuzzy set of real vector functions rather than a fuzzy vector function. In order to solve fuzzy differential equations with fuzzy coefficients, fuzzy initial values and fuzzy forcing functions, Akin et al. (2013) proposed a new algorithm based on an analysis of the crisp solution.

INTRODUCTION:

In this study, we develop numerical methods for nonlinear fuzzy differential equations by an application of the Leapfrog which was studied by Sekar and team of his researchers (2014). Recently, M. Z. Ahmad et al. (2013) and M. Rostamiet al. (2011) discussed the nonlinear fuzzy systems by extension principle and second order Runge-Kutta method. In this paper, the same nonlinear fuzzy differential equations are considered but a different approach using the Adomian decomposition method with more accuracy is presented.

FUZZY SET THEORY

BASIC DEFINITIONS 1.1

In crisp set theory each element in the Universe D is either in or out of A "where $A \subseteq D$

characteristic function is used to characterize any crisp set as follows

$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

but in fuzzy set theory where the partial membership is available, we need to generalize the characteristic function to describe the membership grade of each element in D.

Definition 1.2 (Chen G (2000))

If D is a collection of objects denoted generically by x then a Fuzzy set $\tilde{A}$ in D is a set of ordered pairs:

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}(x)} \right) \middle| x \in D \right\}$$

Where $\mu_{\tilde{A}(x)}$ is the membership function of x in $\tilde{A}$ which maps D to [0, 1]

Example 1.1 (Fuller R (1995))

$A = \{1, 2, 3, ..., 10\}$ in crisp set the set $A \subseteq D$ can be represented as $A = \{4, 10\}$ but in fuzzy set theory it is represented as

$$\tilde{A} = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3), (7, 0.0), (8, 0.0), (9, 0.0), (10, 1)\}$$

Remarks 1.1

- Each element of D can be represented with exactly one ordered pair where the first element from and the second element form D the interval [0,1].
- The value zero is used to represent complete non-membership, the value one is used represent complete membership, and values in between are used to represent intermediate degree of membership.

Definition 1.3 (LeondesC(1998))

The set of elements that belong to Fuzzy set $\tilde{A}$ where its membership degree is at least α is called the" α *level set*":

$$\tilde{A}_\alpha = \left\{ x \in D \middle| \mu_{\tilde{A}(x)} \geq \alpha \right\}$$

Definition 1.4(Leondes C(1998))

The set of elements that belong to Fuzzy set $\tilde{A}$ where its membership degree is greater than α is called the" *strong α level set*": $\tilde{A}_{\alpha} = \left\{ x \in D \middle| \mu_{\tilde{A}(x)} > \alpha \right\}$

Definition 1.5 **(**Ross T. J.,(2010)**)**

Let $\tilde{A} \subseteq D$ then the support of $\tilde{A}$, $S\left(\tilde{A}\right)$ is a crisp set of all elements $x \in D$ such that $\mu_{\tilde{A}(x)} > 0$ $S\left(\tilde{A}\right) = \left\{ x \in D \middle| \mu_{\tilde{A}(x)} > 0 \right\}$

Definition 1.6(Ross T. J.,(2010))

Let $\tilde{A} \subseteq D$ then the kernel of $\tilde{A}$, $\ker\left(\tilde{A}\right)$ is a crisp set of all elements $x \in D$ such that $\mu_{\tilde{A}(x)} = 1, \ker\left(\tilde{A}\right) = \left\{ x \in D \middle| \mu_{\tilde{A}(x)} = 1 \right\}$

Definition 1.7 **(**ZadahL(1965)**)**

A Fuzzy set $\tilde{A}$ is said to be empty iff its membership function is identically zero on D $\mu_{\tilde{A}}(x) = 0, \forall x \in D$

Definition 1.8(ZadahL(1965)**)**

Let $\tilde{A}, \tilde{B} \subseteq D$ then $\tilde{A} \subseteq \tilde{B}$ iff $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in D$

Example 1.2 **(**Fuller R (1995)**)**

Let D and $\tilde{A}$ as in example (1.1), then

$$A_{0.5} = \{2,3,4,5,10\}$$
$$A_{0.5} = \{3,4,5,10\}$$
$$S\left(\tilde{A}\right) = \{1,2,3,4,5,6,10\}$$
$$\ker\left(\tilde{A}\right) = \{4,10\}$$

Convexity is an interesting property. To define it in Fuzzy set theory, we will take the membership function as a reference.

Definition 1.9 (Feng G (2010))

A Fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}\left(\lambda x_1+(1-\lambda)x_2\right)\geq \min\left(\mu_{\tilde{A}}(\lambda x_1),\mu_{\tilde{A}}(\lambda x_2)\right)$

Where $x_1,x_2\in D,\lambda\in[0,1]$

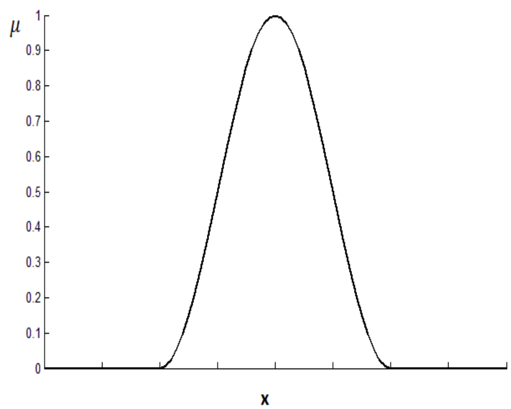

Fig. 1.1 Convex Fuzzy set

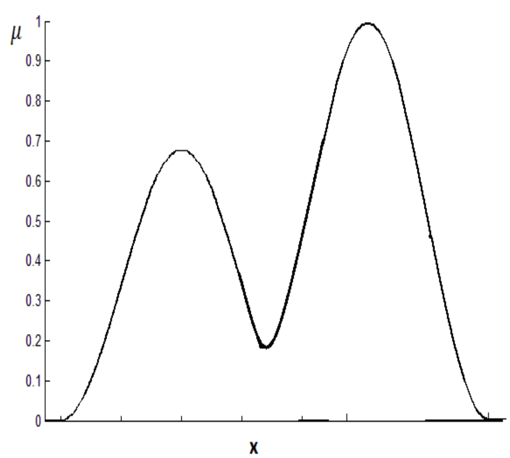

Fig. 1.2 Nonconvex Fuzzy set

Definition 1.10 (Wang L (1997))

Let a finite Fuzzy set $\tilde{A} \subseteq D$ then the cardinality of $\tilde{A}'\left|\tilde{A}'\right|$ is defined as

$\left|\tilde{A}\right| = \sum_{x\in D} \mu_{\tilde{A}}(x)$ and the relative cardinality of $\tilde{A}'\left\|\tilde{A}\right\|$ is defined as $\left\|\tilde{A}\right\| = \dfrac{\left|\tilde{A}\right|}{\left|D\right|}$

1.1Basic operations

Since the membership function is a crucial component of the Fuzzy set theory, it is normal to define Fuzzy set operations via their membership functions.

Definition 1.11 (Zadah L (1965))

Let the Fuzzy sets $\tilde{A}, \tilde{B} \subseteq D$, then the intersection is defined by

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, x \in D \qquad (1.1)$$

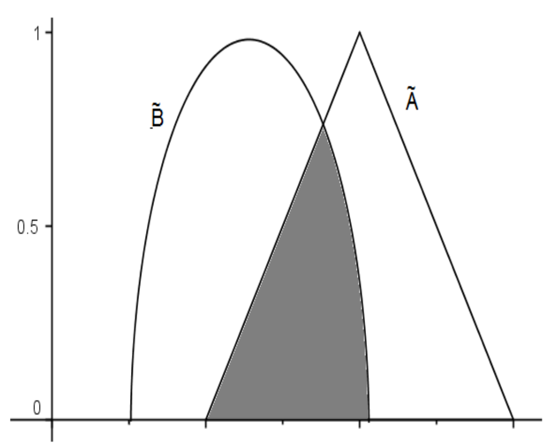

Fig. 1.3 Graphical presentation of $\mu_{\tilde{A}\cap\tilde{B}}(x)$

Definition 1.12 (Zadah L (1965))

Let the Fuzzy sets $\tilde{A}, \tilde{B} \subseteq D$, then the union is defined by

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, x \in D \qquad (1.2)$$

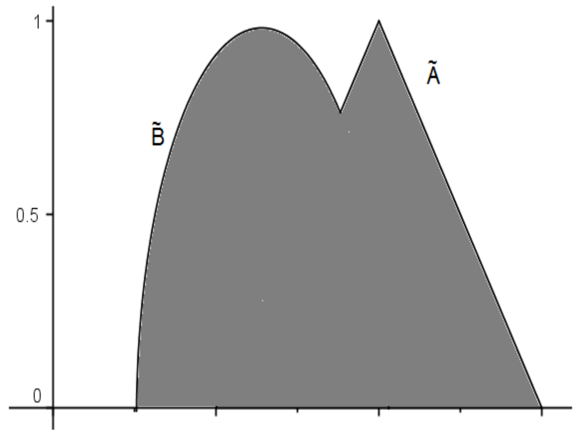

Fig.1.4 Graphical presentation of $\mu_{\tilde{A}\cup\tilde{B}}(x)$

Definition 1.13 (Ross T. J (2010))

Let the Fuzzy sets $\tilde{A}, \tilde{B} \subseteq D$, then the membership function of complement of $\tilde{A}$, is defined by $\mu_{\not\subset\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$ (1.3)

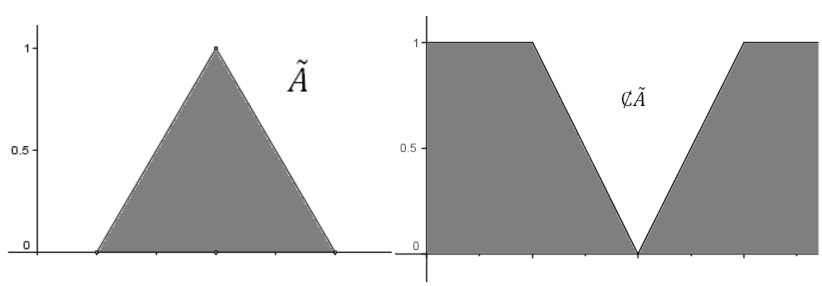

Fig. 1.5 Graphical presentation of $\mu_{\not\subset\tilde{A}}(x)$

Example 1.3(Fuller R. (1995))

Let the Fuzzy sets $\tilde{A}, \tilde{B} \subseteq D$, where D

$$\tilde{A} = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.3),(7,0.0),(8,0.0),(9,0.0),(10,1)\}$$

and

$$\tilde{B} = \{(1,0),(2,0),(3,0.2),(4,0.4),(5,0.6),(6,0.8),(7,1),(8,1),(9,0),(10,0)\}$$

$$\tilde{A} \cup \tilde{B} = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.8),(7,1),(8,1),(9,0),(10,0)\}$$

$$\tilde{A} \cap \tilde{B} = \{(1,0),(2,0),(3,0.2),(4,0.4),(5,0.6),(6,0.3),(7,0),(8,0),(9,0),(10,0)\}$$

$$\not\subset \tilde{B} = \{(1,1),(2,1),(3,0.8),(4,0.6),(5,0.4),(6,0.2),(7,0),(8,0),(9,1),(10,1)\}$$

The operations on Fuzzy sets which are listed as equations (1.1) to (1.3) are called the standard fuzzy operations. These operations are the same as those for classical sets. When the range of membership values is restricted to the unit interval, these standard fuzzy operations are not the only operations that can be applied to Fuzzy sets. For each one of the standard operations, there exists a broad class of functions whose members can be considered a fuzzy generalization of the standard operations. Functions that qualify as fuzzy intersections and fuzzy unions referred to as t-norms and t-co norms respectively.

Definition 1.14 (Ross T. J, (2010))

A mapping $T:[0,1]\times[0,1]\rightarrow[0,1]$ is called t-norm iff $\forall\, x, x', y \; and \; y' \in [0,1]$ T satisfies the properties

$T(x,y) = T(y,x)$ "symmetricity"

$T(x,T(y,z)) = T(T(x,y),z)$ "associatively"

$T(x,y) \le T(x',y') \; if \; x \le x' \; and \; y \le y'$ "monotonicity"

$T(x,1) = x$ "oneidentity"

Definition 1.15(Ross T. J, (2010))

A mapping $S:[0,1]\times[0,1]\rightarrow[0,1]$ is called t-norm iff $\forall\, x, x', y$ *and* $y' \in [0,1]$ S satisfies the properties

$S(x,y)=S(y,x)$ "symmetricity"

$S(x,S(y,z))=S(S(x,y),z)$ "associatively"

$S(x,y)\le S(x',y')$ *if* $x\le x'$ *and* $y\le y'$ "monotonicity"

$S(x,1)=x$ "oneidentity"

Definition 1.16(Leondes C (1998))

Let $\tilde{A}\subseteq x, \tilde{B}\subseteq y$ then the fuzzy Cartesian product $\tilde{A}\times\tilde{B}\subseteq x\times y$ is a Fuzzy set with the following membership function $\mu_{\tilde{A}\times\tilde{B}}(x,y)=\min\left(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\right)$

1.4 Fuzzy relations and its operations

A Fuzzy relation is a mapping from the Cartesian space $x\times y$ to the interval [0, 1], where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes, or $\mu_R(x,y)$.

Definition 1.17(Zimmermann (2001)**)**

Let and be Fuzzy relations on the Cartesian space $x\times y$, then the following operations apply for the membership values set operations.

Union

$\mu_{R\cup G}(x,y)=\max\left(\mu_R(x,y),\mu_G(x,y)\right)$

Intersection

$\mu_{R\cap G}(x,y)=\min\left(\mu_R(x,y),\mu_G(x,y)\right)$

Complement

$\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

Containment

$R \subseteq G \Rightarrow \mu_R(x, y) \leq \mu_G(x, y)$

Definition 1.18(Zimmermann (2001))

Let the Fuzzy relations $R \subseteq x \times y, G \subseteq y \times z$, then sup-min composition of R and G, defined by $(R \circ G)(x, z) = \sup \min \{R(x, y), G(y, z)\}$

The next definition will generalize the Fuzzy relations composition by using any t-norm operation, which is called sup-T composition.

Definition 1.19(Zimmermann (2001))

Let T be a t-norm and let the Fuzzy relations $R \subseteq x \times y, G \subseteq y \times z$, then sup-T composition of R and G, defined by $(R \circ_T G)(x, z) = \sup T \{R(x, y), G(y, z)\}$

The next definition will generalize the Fuzzy relations composition by using any t-norm operation, which is called sup-T composition.

1.5 Fuzzymodeling

Fuzzy models or Fuzzy systems are rule based originating from the concepts of Fuzzy sets, Fuzzy rules, and Fuzzy reasoning. A typical Fuzzy system basically consists of four components: Fuzzy-rule base, inference engine, *fuzzification interface*, and *defuzzification interface*. Fig.(1.6) shows the block diagram of the structure of Fuzzy model.

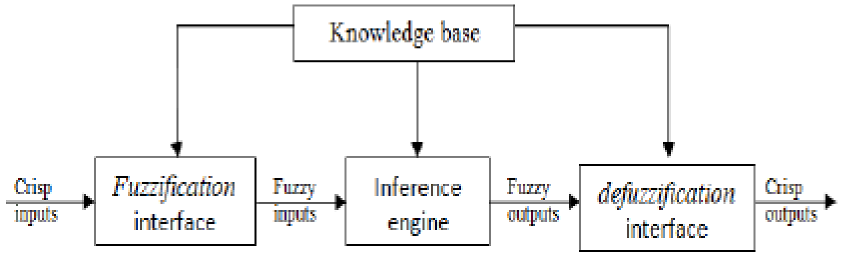

Fig.1.6 Basic structure of mamdani fuzzy model

The inference engine is the central component of Fuzzy model. It is a reasoning mechanism which performs the inference procedure upon the Fuzzy rules and given conditions to derive conclusions. *The fuzzification interface* is a mechanism to transform a real-valued variable to a Fuzzy set. *The defuzzification interface* is a mechanism to transform a Fuzzy set over an output universe of discourse to a real-valued variable. There are many types of Fuzzy model. F-transform is belonging to Takagi-Sugeno models of the 0-th order. It is a fuzzy approximation method based on two transforms: a direct one "fuzzification" and an inverse one "defuzzification". It deals with a fuzzy partition of the domain D given by Fuzzy sets called basic functions, $A_i \subset D, i = 1, 2, ..., n$ fulfilling several conditions.

Definition 1.20(Leondes C (1998))

Let for $i = 1, 2, ..., n, A_i \subset D$ " $A_i \neq \emptyset, A_i \neq x$ ", then the n-tuple $(A_1, A_2, ..., A_n)$ of Fuzzy sets is called a fuzzy partition of X, iff $\sum_{i=1}^{n} \mu_{A_i}(x) = 1$

Conclusion

In this chapter, wehave presentedthe basic concept, definitions and theorems fuzzy sets and fuzzy differential equations.

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