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# Concept of Fuzzy Differential Equations

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## Abstract:

A fuzzy set**A** on **X** is characterized by assigning the grade of belonging to **A** to each element *x* in **X**. The concept of fuzzy numbers and arithmetic operations on it was introduced by Zadeh (1972) Arithmetic structure for fuzzy numbers was later developed by Dubois and Prade (1982) and also Ralescu (1983) all of whom observed the fuzzy number as a location of $\alpha$ levels $0 \le \alpha \le 1$ The fuzzy number $X \in E^n$ is called pyramidal if its $\alpha$ level sets are n-dimensional rectangles for $0 \le \alpha \le 1$ The concept of fuzzy derivative was introduced for the first time by Chang and Zadeh in 1972. Ten years later, Dubois and Prade followed up by making use of the extension principle. Puri and Ralescu (1983) used the notion of H-differentiability to extend the differential of set valued functions to that of fuzzy functions. This helped Seikkala (1987) to introduce the notion of fuzzy derivative as an extension of Hukuhara derivative and the fuzzy integral, which was the same as that proposed by Dubois and Prade (1982).

Kaleva (1987) elaborated the properties of differentiable fuzzy valued mappings by means of the concept of differentiability introduced by Puri and Ralescu (1987). Dubois and Prade (1986) introduced the concept of integration of fuzzy functions for the first time. Goetschel and Voxman (1987) suggested alternative approaches later. Kaleva (1988) defined the integral of fuzzy function, using Lebesgue type concept for integration. The first application of fuzzy integration was put forward by Wu and Ma who made investigations on fuzzy Fredholm integral equation of second kind. Seikkala (1988) defined fuzzy integral exactly as that proposed by Dubois and Prade (1989) and Kaleva (1989) studied measurability and integrability for the fuzzy set valued mappings of a real variable whose values are in the fuzzy number space ($E^n$, D) are established under the compactness type conditions. Currently the fuzzy theory is being developed in two directions. Fuzzification of classical mathematical structures is the first direction while the

second is the practical applications in different areas.

Integral equations are found to occur naturally in many fields of mathematical physics and mechanics. They are also found widely in solutions of differential equations as representation formula. Integral equations can replace differential equations by making them satisfy the boundary conditions. In such cases each solution of integral equation will satisfy the boundary conditions. Many branches of pure analysis such as the theories of functional analysis make use of Integral equations as an efficient tool. A higher dimension to the above applications of Integral equation theory is displayed in the use of fuzzy Integral equations.

**Introduction:**

Fuzzy differential equations are being used for modeling dynamical systems in which vagueness or uncertainties exist. Fuzzy differential equations can be used to describe a large class of such physically important problems. First order linear and nonlinear fuzzy differential equations are one of the simplest fuzzy differential equations which appear in many applications. The term fuzzy differential equation was introduced in 1978 by A.Kandel. Since then the theory of fuzzy differential equations is found split into two independent branches. First one depends upon the notion of Hukuhara derivative (Kaleva) and the other does not. It is also possible to treat fuzzy differential equations without the introduction of any concept of a fuzzy derivative (Seikkala Initial value problem (1987). Hukuhara approach, differential inclusion, quasi flows, differential equations in metric spaces and the so called level wise approach are the several approaches to find a solution for a fuzzy differential equation. To a certain extent all these approaches use $\alpha$ cuts representation of fuzzy sets and thus the solution they defined is based on this notion too.

Fuzzy differential equations were first formulated by Kaleva (1986) and Seikkala (1987) and simultaneously studied the initial value problem. The fuzzy differential dynamical system was invented by Kaleva and Seikkala in the form of time dependent fuzzy differential equations. Uncertainty management in dynamical systems is widely discussed and becoming popular in artificial intelligence particularly in the fields of qualitative and model based reasoning.

Consider the ordinary crisp differential equation $x'(t) = f(t, x(t)), x(0) = x_0$ in this case everything is very specific and without any chance of uncertainty in determining the values of the variables, constants and the form of the equation. But in reality it may so happen that we may not be able to determine the value of $x(0)$ and the values of parameters of $f(t, x(t))$ very accurately. The measurements may have some fluctuations due to uncertainty. One's job will be more complex, if one wants to have accuracy in measurement. The use of fuzzy set theory lessens the complexity to a manageable limit while allowing some uncertainty to remain with the measurement.

Fixed point theorems for fuzzy mappings which are an important tool for showing existence and uniqueness of solutions to fuzzy differential and integral equations have recently been proved by various authors. Lakshmikantham et al. (2003) have proved the existence of fixed points to fuzzy mappings of fuzzy differential equations.

## PRELIMINARY CONCEPTS

**Definition – 2.1 (fuzzy set)**

A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$. In the pair $(x, \mu_{\tilde{A}}(x)$ the first element $x$ belongs to the classical set $A$ and the second element $\mu_{\tilde{A}}(x)$ belongs to the interval [0, 1], called membership function.

**Definition – 2.2 ($\alpha$-cut of a fuzzy set)**

The $\alpha$-level set (or interval of confidence at level $\alpha$ or $\alpha$-cut) of the fuzzy set $\tilde{A}$ of $X$ is a crisp set $A_\alpha$ that contains all the elements of $X$ that have membership values in $A$ greater than or equal to $\alpha$ that is,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \geq \alpha : x \in X, 0 < \alpha \leq 1\}$$

**Definition – 2.3 (fuzzy number)**

The basic definition of fuzzy number is as follows Seikkala (1987).if we denote the set of all real numbers by R and the set of all fuzzy numbers on R is indicated by $R_F$ then a fuzzy number is mapping such that u: R → [0, 1], which satisfies the following four properties:

1. $u$ is upper semi continuous.
2. $u$ is a fuzzy convex;
   a. $(\lambda x + (1 - \lambda)y) \geq min \{u(x), u(y)\}$ for all $x$, $y \in$ R, $\lambda \in$ [0,1].

3. $u$ is normal; that is, $\exists x_0 \in$ R for which $(x_0) = 1$.
4. Sup $u = \{x \in \mathrm{R} \mid u(x) > 0\}$ is support of $u$ and the closure of (sup $u$) is compact.

**Definition - 2.4 (parametric form of fuzzy number (Ming Ma et al (1999))**

A fuzzy number is represented by an ordered pair of functions $(u_1(\alpha), u_2(\alpha))$,

$0 \leq \alpha \leq 1$ which satisfies the following condition

1. $u_1(\alpha)$ is a bounded left continuous non decreasing function for any $\alpha \in [0, 1]$.
2. $u_2(\alpha)$ is a bounded left continuous non increasing function for any $\alpha \in [0, 1]$.
3. $u_1(\alpha) \leq u_2(\alpha)$ for any $\alpha \in [0, 1]$.

**Note.** If $u_1(\alpha) = u_2(\alpha) = \alpha$, then $\alpha$ is a crisp number.

**CONCLUSION**

In this chapter, we have presentedthe basic concept, definitions and theorems fuzzy sets and fuzzy differential equations.

(Gauss Theorem, complex form) Let $D \subset \mathbb{C}$ be a regular domain, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C}), z = x + iy$, then

$$\int_D w_{\bar z}(z)\,dxdy = \frac{1}{2i}\int_{\partial D} w(z)\,dz \quad (2.2)$$

And

$$\int_D w_z(z)\,dxdy = -\frac{1}{2i}\int_{\partial D} w(z)\,d\bar z \quad .(2.3)$$

From the Gauss theorem the Cauchy-Pompeiu representation formulas can be derived.

**THEOREM 3:**

(Cauchy-Pompeiu representations)Let $D \subset \mathbb{C}$ be a regular domain of $\mathbb{C}$,

$$w \in C^1(D;\mathbb{C}) \cap (\overline{D};\mathbb{C}),\ \zeta = \xi + i\eta.$$

Then

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} \quad (2.4)$$

And

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar\zeta}{\bar\zeta - \bar z} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\bar\zeta - \bar z} \quad (2.5)$$

hold for all $z \in D$.

Let us define the integral operator, which is used to solve boundary value problems for the inhomogeneous Cauchy-Riemann equation.

**DEFINITION 4:**

For $f \in L_1(D;\mathbb{C})$ the integral operator

$$T f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C},$$

(2.6)

is called Pompeiu operator.

The Pompeiu operator possesses some important properties listed below.

**RESULTS:**

Let $D \subset \mathbb{C}$ be a bounded domain. If $f \in L_1(D;\mathbb{C})$ then $T f$, is analytic in $\mathbb{C} \setminus \overline{D}$, vanishing at infinity.

**BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS**

Let $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ be the concentric ring domain with the center at the origin.

To solve boundary value problems for analytic functions in R the following representation formula, analogous to (2.13) for the unit disk, is important.

**THEOREM :**

Let $w$ be an analytic function in $R$, continuous on $\overline{R}$. Then there presentation formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} \operatorname{Re} w(z)\left[\frac{\zeta+z}{\zeta-z} + 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{|\zeta|=r}\overline{w(\zeta)}\frac{d\zeta}{\zeta}$$

(2.14)

holds.

**PROOF:**

By the Cauchy theorem we have for any fixed $z \in R$ and any $n \in \mathbb{N}$

$$\frac{1}{2\pi i}\int_{\partial R} w(z)\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\frac{d\zeta}{\zeta} = 0$$

(2.15)

$$\frac{1}{2\pi i}\int_{\partial R} w(z)\left[\frac{\overline{z}\zeta}{1-\overline{z}\zeta} - \sum_{n=1}^{\infty}\left(\frac{r^{2n}\overline{z}\zeta}{r^{2n}\overline{z}\zeta - 1} + \frac{r^{2n}}{\overline{z}\zeta - r^{2n}}\right)\right]\frac{d\zeta}{\zeta} = 0$$

(2.16)

By adding (2.15) and the complex conjugate of (2.16) to the right-hand side of the Cauchy formula, applied to $w$ in R.

We get

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial R} w(\zeta)\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial R}\overline{w(\zeta)}\left[\frac{z\overline{\zeta}}{1 - z\overline{\zeta}} - \sum_{n=1}^{\infty}\left(\frac{r^{2n}\overline{\zeta}}{r^{2n}\overline{\zeta} - 1} + \frac{r^{2n}}{\overline{\zeta} - r^{2n}}\right)\right]\frac{d\zeta}{\zeta}.$$

Or

$$w(z)=\frac{1}{2\pi i}\int_{\partial D}\operatorname{Re} w(\zeta)\left[\frac{\zeta}{\zeta-z}+\frac{z|\zeta|^2}{\zeta-z|\zeta|^2}+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2-\zeta}-\frac{r^{2n}\zeta}{z|\zeta|^2-r^{2n}\zeta}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial R}\operatorname{Im} w(\zeta)\left[\frac{\zeta}{\zeta-z}-\frac{z|\zeta|^2}{\zeta-z|\zeta|^2}+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2-\zeta}+\frac{r^{2n}\zeta}{z|\zeta|^2-r^{2n}\zeta}\right)\right]\frac{d\zeta}{\zeta}$$

Dividing the boundary into the two components and performing some simplifications lead to,

$$w(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{|\zeta|=r}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+1+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int_{|\zeta|=1}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta},$$

Which is equivalent to

$$w(z)=\frac{1}{2\pi i}\int_{\partial R}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int_{\partial R}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int_{|\zeta|=r}\overline{w(\zeta)}\frac{d\zeta}{\zeta}.$$

Then the result follows if one takes into account that $\frac{1}{2\pi i}\int_{\partial R} w(\zeta)\frac{d\zeta}{\zeta}=0$ for $w$ being analytic in $R$ and therefore $\frac{1}{2\pi}\int_{\partial R}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}=0.$

**Conclusion:**

In this chapter we studied about boundary value problem for first order complex partial differential equations and second order complex partial differential equations in a ring domain. It helps us to solve the boundary valued problem for analytic functions, and green function for a circular ring domain and also to know about conjugate of complex number. Boundary value problems for the inhomogeneous construction of Cauchy - riemann equation, the biharmonic green function of a circular ring domain.

**Reference:-**

01. Abdymanapov S.A., Begehr H., Tungatarov A.B. Some Schwarz

problems in a quarter plane. Eurasian Math. J. 2005. No.3. P. 22 - 35.

02. Adler P.M. Porous Media. Geometry and Transport. - Paris: Butterworth/ Heinemann, 1992.-151 p.

03. Akhiezer N.I. Elements of the Theory of Elliptic Functions. Rhode Island: Providence, 1990 - 238 p.

04. Almansi E. Sull'integrazionedell'equatzionedifferenziale $\Delta^{2n} u = 0$. Ann. Mat. - 1899. No. 2(3). P. 1-9.