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Boundary Value Problems for First Order Complex Partial Differential Equations in a Ring Domain

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Abstract:

The theory of boundary value problems for complex partial differential equation combines knowledge and methods from many fields of mathematics, i.e. complex analysis, partial differential equations, functional analysis, equations of mathematical physics etc.

One of the main aim of the theory of complex boundary value problems is to obtain solutions in analytic or closed form. Many results in this directions are known for special kinds of equations, namely, for the Cauchy-Riemann equation, the Beltrami equation, for elliptic equations with constant or analytic coefficients etc. Boundary value problems were mostly considered in simply connected domains, i.e. unit disk, a half plane, a corner etc. The intention to investigate boundary value problems for multiply connected domains gives rise to many additional difficulties even in the simplest case dealing with analytic functions. There are just few results known on this subject. Among them is the celebrated Villat's formula for the solution of Schwarz problem in a concentric ring domain, expressed in terms of Weierstrass ζ -function and the formulas obtained. As well for the Schwarz and the Riemann-Hilbert problems as for the Riemann problem of linear conjugacy for a multiply connected circular domain in form of series with respect to the elements of a special Schottky symmetry group. The main difficulty appeared is connected with the single validness of solutions while passing for simply to multiply connected domains.A systematic investigation of boundary value problems for complex partial differential equations of arbitrary order on the base of integral representation formulas was initiated by H. Begehr. To start with, the basic boundary value problems for model equations are observed. The differential operator of a model equation consists of a product of powers of the complex Cauchy-Riemann operator $\partial\bar{z}$ and its complex conjugate. The main methods of the theory will be pointed on now. The complex form of the Gauss theorem for a regular domain D on the complex plane $\mathbb{C}$ and an arbitrary function $w \in c^1(D;\mathbb{C}) \cap (\bar{D};\mathbb{C})$ leads to the Cauchy-Pompeiu representation formula. This formula is a generalization of the Cauchy formula for analytic functions. The area integral appearing in the Cauchy-Pompeiu formula is called the Pompeiu operator. It plays an important role in treating boundary value problems for complex partial differential equations. The properties of the Pompeiu operator were studied by I.N. Vekua. If f belongs to $L_p(D;\mathbb{C})$, $p > 1$, then Tf possesses weak derivatives with respect to z and $\bar{z}$,moreover $\partial\bar{z}\, Tf = f$, $\partial z Tf =: \Pi f$, where Π is a singular integral being understood in the principle value sense. Integrals of such type are investigated.

Introduction:

Four basic boundary value problems, namely, the Schwarz, the Dirichlet, the

Neumann, the Robin problems for analytic functions and more generally for the inhomogeneous Cauchy-Riemann equation are investigated in a concentric ring domain. The representations for the solutions and solvability conditions are given in explicit form.

NOTATIONS AND TECHNICAL PRELIMINARIES

Let $\mathbb{C}$ be the complex plane of the variable $z = x + iy,\ \ y \in \mathbb{R}$. The extended complex plane is denoted by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The complex number $\overline{z} = x - iy$ is called the conjugate number to z. By Re z Im z the real and imaginary part of z are denoted.

The complex partial differential operators of first order are defined by

$$\partial_z = \frac{1}{2}(\partial x - i\partial y),\ \partial_{\overline{z}} = \frac{1}{2}(\partial x + i\partial y).$$

(2.1)

A complex-valued function $w = u + iv$ is given by a couple of real-valued functions $u = u(x, y), v = v(x, y)$ and being a function of the two variables z and $\overline{z}$. In the case, when u and v are differentiable and w is independent of $\overline{z}$ in an open set of the complex plane, the function w is said to be analytic in the set; the functions u, v then satisfy the Cauchy-Riemann system of partial differential equations

$$\partial_x u = \partial_y v,\ \ \partial_y u = -\partial_x v,$$

which is equivalent to the complex homogeneous Cauchy-Riemann equation

$$\partial_{\overline{z}} w = 0.$$

For analytic functions the Cauchy theorem is valid.

THEOREM 1:

Let w be an analytic function in a simply connected domain $D \subset \mathbb{C}$ and let Γ be a simple closed smooth curve, $\Gamma \subset D$ .Then

$$\int_{\Gamma} w(z)\, dz = 0.$$

From the Cauchy theorem the representation of an analytic function via the Cauchy type integral is deduced. A simple closed smooth curve Γ on the complex plane divides the plane in to two parts $\operatorname{int}\Gamma,\ ext\,\Gamma$ which are internal and external domains with respect to Γ. If w is analytic in $\operatorname{int}\Gamma$ and continuous in $\overline{\operatorname{int}\Gamma}$, then

$$\frac{1}{2\pi i}\int_{\Gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(z), & z \in \operatorname{int}\Gamma, \\ 0, & z \in \operatorname{int}\Gamma. \end{cases}$$

If w is analytic in $ext\,\Gamma$ and continuous in $\overline{ext\,\Gamma}$, then

$$\frac{1}{2\pi i}\int_{\Gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(\infty), & z \in \operatorname{int}\Gamma, \\ -w(z) + w(\infty), & z \in \operatorname{int}\Gamma. \end{cases}$$

A domain D on the complex plane is said to be regular if it is bounded and its boundary ∂D is smooth.

The fundamental tools for solving boundary value problems for complex first order partial differential equations are the Gauss theorem and the Cauchy-Pompeiu representation formulas.

THEOREM 2:

(Gauss Theorem, complex form) Let $D \subset \mathbb{C}$ be a regular domain, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C}), z = x + iy,$ then

$$\int_D w_{\overline{z}}(z)\, dxdy = \frac{1}{2i}\int_{\partial D} w(z)\, dz \qquad (2.2)$$

And

$$\int_D w_z(z)\, dxdy = -\frac{1}{2i}\int_{\partial D} w(z)\, d\overline{z} \qquad .(2.3)$$

From the Gauss theorem the Cauchy-Pompeiu representation formulas can be derived.

THEOREM 3:

(Cauchy-Pompeiu representations)Let $D \subset \mathbb{C}$ be a regular domain of $\mathbb{C}$,

$w \in C^1(D;\mathbb{C}) \cap (\overline{D};\mathbb{C})$, $\zeta = \xi + i\eta$.

Then

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} \quad (2.4)$$

And

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\bar{\zeta} - \bar{z}} \quad (2.5)$$

hold for all $z \in D$.

Let us define the integral operator, which is used to solve boundary value problems for the inhomogeneous Cauchy-Riemann equation.

DEFINITION 4:

For $f \in L_1(D;\mathbb{C})$ the integral operator

$$T f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C}, \quad (2.6)$$

is called Pompeiu operator.

The Pompeiu operator possesses some important properties listed below.

RESULTS:

Let $D \subset \mathbb{C}$ be a bounded domain. If $f \in L_1(D;\mathbb{C})$ then $T f$, is analytic in $\mathbb{C} \setminus \overline{D}$, vanishing at infinity.

BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS

Let $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ be the concentric ring domain with the center at the origin.

To solve boundary value problems for analytic functions in R the following representation formula, analogous to (2.13) for the unit disk, is important.

THEOREM :

Let w be an analytic function in R, continuous on $\overline{R}$. Then there presentation formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} \operatorname{Re} w(z)\left[\frac{\zeta + z}{\zeta - z} + 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{|\zeta|=r} \overline{w(\zeta)}\frac{d\zeta}{\zeta} \quad (2.14)$$

holds.

PROOF:

By the Cauchy theorem we have for any fixed $z \in R$ and any $n \in \mathbb{N}$

$$\frac{1}{2\pi i}\int_{\partial R} w(z)\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\frac{d\zeta}{\zeta} = 0 \quad (2.15)$$

$$\frac{1}{2\pi i}\int_{\partial R} w(z)\left[\frac{\bar{z}\zeta}{1 - \bar{z}\zeta} - \sum_{n=1}^{\infty}\left(\frac{r^{2n}\bar{z}\zeta}{r^{2n}\bar{z}\zeta - 1} + \frac{r^{2n}}{\bar{z}\zeta - r^{2n}}\right)\right]\frac{d\zeta}{\zeta} = 0 \quad (2.16)$$

By adding (2.15) and the complex conjugate of (2.16) to the right-hand side of the Cauchy formula, applied to w in R.

We get

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial R} w(\zeta)\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\right)\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial R} \overline{w(\zeta)}\left[\frac{z\bar{\zeta}}{1 - z\bar{\zeta}} - \sum_{n=1}^{\infty}\left(\frac{r^{2n}\bar{\zeta}}{r^{2n}\bar{\zeta} - 1} + \frac{r^{2n}}{\bar{\zeta} - r^{2n}}\right)\right]\frac{d\zeta}{\zeta}.$$

Or

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} \operatorname{Re} w(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{z|\zeta|^2}{\zeta - z|\zeta|^2}\right.$$

$$\left.+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} - \frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2 - \zeta} - \frac{r^{2n}\zeta}{z|\zeta|^2 - r^{2n}\zeta}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial R}\operatorname{Im} w(\zeta)\left[\frac{\zeta}{\zeta-z}-\frac{z|\zeta|^2}{\zeta-z|\zeta|^2}+\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2-\zeta}+\frac{r^{2n}\zeta}{z|\zeta|^2-r^{2n}\zeta}\right)\right]\frac{d\zeta}{\zeta}.$$

Dividing the boundary into the two components and performing some simplifications lead to,

$$w(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{|\zeta|=r}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+1+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int_{|\zeta|=1}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta},$$

Which is equivalent to

$$w(z)=\frac{1}{2\pi i}\int_{\partial R}\operatorname{Re} w(\zeta)\left[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\right)\right]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi}\int_{\partial R}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int_{|\zeta|=r}\overline{w(\zeta)}\frac{d\zeta}{\zeta}.$$

Then the result follows if one takes into account that $\frac{1}{2\pi i}\int_{\partial R} w(\zeta)\frac{d\zeta}{\zeta}=0$ for w being analytic in R and therefore $\frac{1}{2\pi}\int_{\partial R}\operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta}=0.$

Conclusion:

In this chapter we studied about boundary value problem for first order complex partial differential equations and second order complex partial differential equations in a ring domain. It helps us to solve the boundary valued problem for analytic functions, and green function for a circular ring domain and also to know about conjugate of complex number. Boundary value problems for the inhomogeneous Cauchy - riemann equation,construction of the biharmonic green function of a circular ring domain.

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